Null-Pole Subspaces of Nonregular Rational Matrix Functions

Joseph A. Ball* and Marek Rakowski†
Department of Mathematics
Virginia Tech
Blacksburg, Virginia 24061

Submitted by Leiba Rodman

ABSTRACT

Given a (not necessarily regular) rational matrix function W and a subset σ of the extended complex plane, we associate with it a free module $\mathcal{S}_\sigma(W)$ over the ring of scalar rational functions which are analytic on σ, called the null-pole subspace of W over σ. In the scalar case, the information encoded in this module is equivalent to the knowledge of all the poles and zeros, including their multiplicities, at all points of σ; in the general matrix case, the null-pole subspace encodes the more complicated zero-pole structure of a rational matrix function, which is the key tool for the understanding of many factorization problems. In this paper we show how various other modules which have been introduced in the literature in connection with pole-zero structure (e.g. pole module, zero module) can be read off from the null-pole subspace, and how, conversely, the null-pole subspace can be recovered from its various pieces (pole module, null module, a null-pole coupling operator and left annihilator). We give an analytic, coordinate-dependent description of these modules which has connection with realization theory and which is useful for computations. We also solve the converse problem of finding a rational matrix function which has a given admissible module as its null-pole subspace.

---

0. INTRODUCTION

Let W_1 and W_2 be rational matrix functions, and let σ be a subset of the extended complex plane $\mathbf{C}_\infty$. A natural problem connecting W_1 and W_2 is to decide whether there exists a rational matrix function Q analytic on σ such

---

*The research of this author was partially supported by National Science Foundation Grant DMS 870165-02.

†The research of this author was partially supported by the Institute of Electron Technology CEMI, Warszawa, Poland.

© Elsevier Science Publishing Co., Inc., 1991
655 Avenue of the Americas, New York, NY 10010 0024-3795/91/$3.50

that $W_1 = W_2Q$. For example, if $\sigma = \{\infty\}$, this is the causal factorization problem investigated in [16]. The problem has a simple translation into the language of modules. Let $\mathscr{R}$ be the field of scalar rational functions. If σ is a subset of $\mathbf{C}_\infty$, let $\mathscr{R}(\sigma)$ denote the subring of $\mathscr{R}$ formed by members of $\mathscr{R}$ which are analytic on σ. Let $\mathscr{R}^{m\times n}$ and $\mathscr{R}^{m\times n}(\sigma)$ denote the free $\mathscr{R}$ and $\mathscr{R}(\sigma)$ modules of $m \times n$ matrices with entries in $\mathscr{R}$ and $\mathscr{R}(\sigma)$, and suppose $W_1 \in \mathscr{R}^{m\times n_1}$ and $W_2 \in \mathscr{R}^{m\times n_2}$. Then $W_1 = W_2Q$ for some $Q \in \mathscr{R}^{n_2\times n_1}(\sigma)$ if and only if $W_1\mathscr{R}^{n_1\times 1}(\sigma) \subset W_2\mathscr{R}^{n_2\times 1}(\sigma)$.

Since $\mathscr{R}^{m\times 1}$ is a vector space over $\mathscr{R}$, and $\mathscr{R}(\sigma)$ is a subring of $\mathscr{R}$, $W\mathscr{R}^{n\times 1}(\sigma)$ is an $\mathscr{R}(\sigma)$-module. In particular, it is a vector space over $\mathbf{C}$. The space $W\mathscr{R}^{n\times 1}(\sigma)$ is called the *left null-pole subspace* of W over σ. We shall denote this space by $\mathscr{S}_\sigma(W)$. For the regular case, that is, when the involved functions are square and have determinants which do not vanish identically, null-pole subspaces have played a key role in [13, 14, 5, 2] (see [6] for a comprehensive exposition). The main point is that the null-pole subspace of a rational matrix function W has an explicit description in terms of null and pole data of W. If a function $W \in \mathscr{R}^{m\times m}$ is regular and $\sigma \subset \mathbf{C}$, the left null-pole subspace for W over σ is determined by a right pole pair (C_π, A_π) for W over σ, a left null pair (A_ζ, B_ζ) for W over σ (see [3]), and a matrix Γ which *couples* the pairs (C_π, A_π) and (A_ζ, B_ζ). The explicit representation of $W\mathscr{R}^{m\times 1}(\sigma)$ in terms of (C_π, A_π), (A_ζ, B_ζ), and Γ is given by the formula (see Theorem 3.4.1. in [4])

$$W\mathscr{R}^{m\times 1}(\sigma) = \Big\{ C_\pi(z - A_\pi)^{-1}x + h(z) : x \in \mathbf{C}^{n_\pi\times 1},\ h \in \mathscr{R}^{m\times 1}(\sigma),$$

$$\text{and } \sum_{z_0\in\sigma} \operatorname{Res}_{z=z_0}(z - A_\zeta)^{-1}B_\zeta h(z) = \Gamma x \Big\},$$

where n_π is such that $A_\pi \in \mathbf{C}^{n_\pi\times n_\pi}$. The matrix Γ above is called a null-pole coupling matrix associated with the right-pole pair (C_π, A_π) and the left-null pair (A_ζ, B_ζ) for W over σ (see [4]), or a coupling operator (see, e.g. [13]). When W has the value I at infinity, a pole pair (C_π, A_π), a null pair (A_ζ, B_ζ), and the associated coupling matrix Γ are easily computed from a minimal realization $W(z) = I + C(z - A)^{-1}B$ for $W(z)$.

In this paper we obtain a similar description for the nonregular case. We build on the results on the zero structure and the pole structure of a nonregular rational matrix function derived in [12]—this paper is a continuation of [12]. In contrast with [12], this paper takes a more module-theoretic approach. In particular, we connect the analytic description of the zero and pole structure of a function $W \in \mathscr{R}^{m\times n}$ in terms of pole pairs, null pairs, and kernel triples with the basis-free description in terms of certain pole and zero modules. Pole and zero modules have played a prominent role for some time in the algebraic approach to systems theory (see e.g. [22] for pole modules, [29] for zero modules, and [26] for an earlier algebraic approach to zeros). If $\sigma = \mathbf{C}$, these modules are isomorphic with the finite pole and zero modules recently studied by Wyman, Sain, Conte, and Perdon (see [31, 17]); for a more general subset σ see also [30]. We also obtain a characterization of which $\mathscr{R}(\sigma)$-submodules of $\mathscr{R}^{m \times 1}$ arise as a null-pole subspace over σ for some rational matrix function. The characterization of which $\mathscr{R}(\sigma)$-submodules of $\mathscr{R}^{m \times 1}$ arise as the null-pole subspace of a regular rational matrix function is presented in Chapter 14 of [6]. In the last section of this paper, we show that the construction of a rectangular matrix function with a given left zero and right pole structure, which has been developed in [12], can be used to produce functions with a given left null-pole subspace over $\sigma \subset \mathbf{C}$.

As was mentioned above, our main motivation for the study of null-pole subspaces comes from applications to factorization theory of rational matrix functions. This connection (in the more general setting of nonrational matrix functions) can be seen in the Beurling-Lax theorem (see [28, 25]), where the inner factor B of an H^{∞} function $F = BG$ is the inner representer for the shift-invariant subspace (i.e. "null-pole subspace") FH^2 generated by F. The connection between representations of shift-invariant subspaces and various types of factorizations (e.g. J-inner-outer, Wiener-Hopf, or spectral), as well as interpolation and dilation problems, was obtained in [7–10]. The program of [13, 14, 2] was to refine these ideas for the regular rational case so as to obtain explicit realization formulas for J-inner-outer, Wiener-Hopf, and coprime factorizations, as well as linear fractional formulas for the set of all solutions of interpolation and dilation problems; all these results have been refined and codified in the recent book [6]. This paper continues the work of [11, 12] to extend this program to the nonregular case. The characterization of the null-pole subspace in terms of null-pole data carried out here is but a part of a general program and turns out to be considerably more technical than in the regular case. We plan to return to other parts of the program in future work and ultimately obtain as applications results on J-inner-outer factorization and matricial Nevanlinna-Pick interpolation problems more general than those considered in [6]. State-space formulas for inner and outer factors of rectangular rational matrix functions have recently been obtained in [15] and [32], but much remains to be understood about interpolation problems of the sort studied in [20].

Below, we shall frequently use the notion of orthogonality in $\mathscr{R}^{m \times 1}$ (or $\mathscr{R}^{1 \times m}$), in the sense of valuation theory, which we now review (see [18, 11]). If $\lambda \in \mathbf{C}_{\infty}$, one can define a non-Archimedean norm $\|\cdot\|_{z=\lambda}$ on $\mathscr{R}^{m \times 1}$ by putting, for nonzero ϕ, $\|\phi\|_{z=\lambda} = e^{\gamma}$ with γ an integer such that $(z - \lambda)^{\gamma}\phi(z)$

[or $z^{-\gamma}\phi(z)$ if $\lambda=\infty$] is analytic, and does not vanish, at λ. We say that subspaces Ω and Λ are *orthogonal* at λ if Ω and Λ are orthogonal with respect to the norm $\|\cdot\|_{z=\lambda}$, that is, if

$$\|x+y\|_{z=\lambda}=\max\{\|x\|_{z=\lambda},\|y\|_{z=\lambda}\}$$

for each $x\in\Omega$ and $y\in\Lambda$ (see [23] for a comprehensive treatment of non-Archimedean normed spaces). If Ξ is a subspace of $\mathscr{R}^{m\times 1}$, let

$$\Xi(\lambda)=\{\phi(\lambda):\phi\in\Xi \text{ and } \phi \text{ is analytic at } \lambda\}.$$

Then $\Xi(\lambda)$ is a subspace of $\mathbf{C}^{m\times 1}$ and $\dim_{\mathbf{C}}\Xi(\lambda)=\dim_{\mathscr{R}}\Xi$. In this notation, Ω and Λ are orthogonal at λ if and only if

$$\Omega(\lambda)\cap\Lambda(\lambda)=(0)$$

(see Proposition 2.3 in [12]). If $\sigma\subset\mathbf{C}_\infty$ and Ω,Λ are orthogonal at every point of σ, we say that Ω,Λ are orthogonal on σ and write $\Omega\oplus_\sigma\Lambda$. If σ is a proper subset of $\mathbf{C}_\infty$, then for every subspace Ω of $\mathscr{R}^{m\times 1}$ there exists an algebraic complement Λ of Ω such that

$$\mathscr{R}^{m\times 1}=\Omega\oplus_\sigma\Lambda \tag{0.1}$$

(see Corollary 2.6 in [12]). We call such Λ an orthogonal complement of Ω in $(\mathscr{R}^{m\times 1},\sigma)$. Usually Ω has many orthogonal complements in $(\mathscr{R}^{m\times 1},\sigma)$. A useful property of the decomposition (0.1) is that

$$\text{if}\quad h\in\mathscr{R}^{m\times 1}(\sigma) \text{ and } h=h_\Omega+h_\Lambda \quad\text{with } h_\Omega\in\Omega \text{ and } h_\Lambda\in\Lambda,$$

$$\text{then } h_\Omega,h_\Lambda\in\mathscr{R}^{m\times 1}(\sigma). \tag{0.2}$$

We conclude the introduction with the remark that the right null-pole subspace of a function $W\in\mathscr{R}^{m\times n}$ over a subset σ of $\mathbf{C}$ is defined in the analogous way. That is, the right null-pole subspace of W over σ is the free $\mathscr{R}(\sigma)$-module $\mathscr{R}^{1\times m}(\sigma)W$, viewed as a vector space over $\mathbf{C}$. All the results on left null-pole subspaces have their analogues for right null-pole subspaces. These analogues can be derived in a similar way or, alternatively, can be obtained by considering the transpose of W. Therefore below we confine our attention exclusively to left null-pole subspaces.

1. BASIS-FREE DESCRIPTION OF NULL-POLE SUBSPACES

Let $W \in \mathscr{R}^{m\times n}$, let σ be a proper subset of $\mathbf{C}_\infty$, and let $\mathscr{S}_\sigma(W) = W\mathscr{R}^{n\times 1}(\sigma)$. We will call the $\mathscr{R}(\sigma)$-module

$$\mathscr{S}_\pi = \frac{\mathscr{S}_\sigma(W)}{\mathscr{S}_\sigma(W)\cap\mathscr{R}^{m\times 1}(\sigma)} \tag{1.1}$$

the *(right) pole module* of W over σ. Plainly, $\mathscr{S}_\pi$ is a torsion module. It can be easily shown that $\mathscr{S}_\pi$ is a finite-dimensional vector space over $\mathbf{C}$, and $\dim_{\mathbf{C}}\mathscr{S}_\pi$ is equal to the sum of the multiplicities of all poles of W in σ. Here the multiplicity of a pole $\lambda\in\mathbf{C}$ of W is the sum of the orders of poles at λ of the diagonal entries in the Smith-McMillan form of W, and the multiplicity of a pole of W at infinity is the multiplicity of the pole of $H(z) = W(z^{-1})$ at 0.

We will call the $\mathscr{R}(\sigma)$-module

$$\mathscr{S}_\zeta = \frac{W\mathscr{R}^{n\times 1}\cap\mathscr{R}^{m\times 1}(\sigma)}{\mathscr{S}_\sigma(W)\cap\mathscr{R}^{m\times 1}(\sigma)} \tag{1.2}$$

a *(discrete left) zero module* of W over σ. It is clear from the definition that $\mathscr{S}_\zeta$ is a torsion module. It is not difficult to show that $\mathscr{S}_\zeta$ is finite-dimensional as a vector space over $\mathbf{C}$. In fact, $\dim_{\mathbf{C}}\mathscr{S}_\zeta$ is equal to the sum of multiplicities of all zeros of W in σ, where the multiplicity of a zero of W at $\lambda\in\mathbf{C}$ is the sum of orders of zeros at λ of the nonzero diagonal entries of the Smith-McMillan form of W, and the multiplicity of a zero of W at infinity is the multiplicity of a zero of $H(z) = W(z^{-1})$ at 0.

The *(left) annihilator* of $\mathscr{S}_\sigma(W)$ is the $\mathscr{R}(\sigma)$-submodule of $\mathscr{R}^{1\times m}$

$$W^{\circ l} = \{f\in\mathscr{R}^{1\times m} : fg = 0\ \forall g\in\mathscr{S}_\sigma(W)\}.$$

In fact, $W^{\circ l}$ is an $\mathscr{R}$-subspace of $\mathscr{R}^{1\times m}$, and so the degrees of vector polynomials in any minimal polynomial basis for $W^{\circ l}$ are determined by $W^{\circ l}$ (see [18]). They are called the *left Forney indices* of W (see [1]). The left annihilator of W determines the column span (over $\mathscr{R}$) of W:

$$W\mathscr{R}^{n\times 1} = \{g\in\mathscr{R}^{m\times 1} : fg = 0\ \forall f\in W^{\circ l}\}. \tag{1.3}$$

This column span is also easily obtained from $\mathcal{S}_\sigma(W)$ as the smallest $\mathcal{R}$-linear subspace containing $\mathcal{S}_\sigma(W)$.

It is clear that $\mathcal{S}_\sigma(W)$ uniquely determines the modules $\mathcal{S}_\pi$, $\mathcal{S}_\zeta$, and $W^{\circ l}$. However, the converse may fail. We show now that $\mathcal{S}_\pi$, $\mathcal{S}_\zeta$, and $W^{\circ l}$ do not determine $\mathcal{S}_\sigma(W)$ completely. To this end, we define first an $\mathcal{R}(\sigma)$-module $\mathcal{P}_W$ which is isomorphic to $\mathcal{S}_\pi$. Let $\mathcal{R}^{m\times 1}_0(\sigma^c)$ be the $\mathbf{C}$-linear subspace of $\mathcal{R}^{m\times 1}$ formed by functions which are analytic in $\mathbf{C}_\infty\setminus\sigma$ and vanish at infinity. By partial-fraction expansions, $\mathcal{R}^{m\times 1}$ has a direct-sum decomposition

$$\mathcal{R}^{m\times 1} = \mathcal{R}^{m\times 1}(\sigma) \dotplus \mathcal{R}^{m\times 1}_0(\sigma^c)$$

where all vector spaces are over $\mathbf{C}$. (Throughout the paper, we will use the symbol "$\dotplus$" to denote a direct-sum decomposition of a vector space over $\mathbf{C}$ into $\mathbf{C}$-linear subspaces.) Let P_{σ^c} be the projection of $\mathcal{R}^{m\times 1}$ onto $\mathcal{R}^{m\times}(\sigma^c)$ along $\mathcal{R}^{m\times 1}(\sigma)$. Let $\mathcal{P}_W = P_{\sigma^c}(\mathcal{S}_\sigma(W))$, and if $h\in\mathcal{P}_W$ and $r\in\mathcal{R}(\sigma)$, let $r\cdot h = P_{\sigma^c}(rh)$. Then $(\mathcal{P}_W,\cdot)$ is an $\mathcal{R}(\sigma)$-module which is isomorphic to $\mathcal{S}_\pi$. In fact, the function $\phi:\mathcal{S}_\pi\to\mathcal{P}_W$ which sends $h+[\mathcal{S}_\sigma(W)\cap\mathcal{R}^{m\times 1}(\sigma)]$ to $P_{\sigma^c}(h)$ is an $\mathcal{R}(\sigma)$-module isomorphism. We note that $(\mathcal{P}_W,\cdot)$ is a $\mathbf{C}$-linear subspace of $\mathcal{R}^{m\times 1}$.

We point out that $\mathcal{S}_\sigma(W)\cap\mathcal{R}^{m\times 1}(\sigma)$ is an $\mathcal{R}(\sigma)$-submodule of $\mathcal{R}^{m\times 1}$ which determines both $\mathcal{S}_\zeta$ and $W^{\circ l}$. Indeed, $W^{\circ l}$ is the $\mathcal{R}$-linear subspace of $\mathcal{R}^{m\times 1}$ formed by elements which annihilate $\mathcal{S}_\sigma(W)\cap\mathcal{R}^{m\times 1}(\sigma)$, and $\mathcal{S}_\zeta$ is determined by (1.3) and (1.2). Conversely, $\mathcal{S}_\sigma(W)\cap\mathcal{R}^{m\times 1}(\sigma)$ is determined by $\mathcal{S}_\zeta$ and $W^{\circ l}$ as the kernel of a canonical map $W\mathcal{R}^{n\times 1}\cap\mathcal{R}^{m\times 1}(\sigma)\to\mathcal{S}_\zeta$.

Let $\mathcal{N}$ be a $\mathbf{C}$-linear subspace of $\mathcal{R}^{m\times 1}(\sigma)$ such that

$$\mathcal{R}^{m\times 1}(\sigma) = \mathcal{N} \dotplus \left[\mathcal{S}_\sigma(W)\cap\mathcal{R}^{m\times 1}(\sigma)\right]. \tag{1.4}$$

The following decomposition of $\mathcal{S}_\sigma(W)$ extends to the nonregular case the decomposition appearing in various forms in [13], [14], and [2].

THEOREM 1.1. *Let* $\sigma\subset\mathbf{C}_\infty$ *and let* $W\in R^{m\times n}$. *Let* $\mathcal{P}_W\approx\mathcal{S}_\pi$ *be as above, and suppose* $\mathcal{N}$ *is such that* (1.4) *holds. Then there exists a unique* $\mathbf{C}$*-linear transformation* $T:\mathcal{P}_W\to\mathcal{N}$ *such that* $(I+T)f\in\mathcal{S}_\sigma(W)$ *for every* $f\in\mathcal{P}_W$. *Moreover,*

$$\mathcal{S}_\sigma(W) = (I+T)\mathcal{P}_W \dotplus \left[\mathcal{S}_\sigma(W)\cap\mathcal{R}^{m\times 1}(\sigma)\right]. \tag{1.5}$$

Proof. The proof is based on a simple geometric argument. Let $x \in \mathscr{P}_W$, and let $s \in \mathscr{S}_\sigma(W)$ be such that $x = P_{\sigma'}(s)$. Let y_x be the projection of $s - x$ onto $\mathscr{N}$ along $\mathscr{S}_\sigma(W) \cap \mathscr{R}^{m\times 1}(\sigma)$. Then $s - x - y_x \in \mathscr{S}_\sigma(W) \cap \mathscr{R}^{m\times 1}(\sigma)$, so $x + y_x = s - (s - x - y_x) \in \mathscr{S}_\sigma(W)$. If $y_1, y_2 \in \mathscr{N}$ and $x + y_1, x + y_2 \in \mathscr{S}_\sigma(W)$, then $y_1 - y_2 \in \mathscr{S}_\sigma(W) \cap \mathscr{N} = \{0\}$. Thus, for every $s \in \mathscr{P}_W$ there is a unique $y_x \in \mathscr{N}$ such that $x + y_x \in \mathscr{S}_\sigma(W)$, and we can define a map $T : \mathscr{P}_W \to \mathscr{N}$ by putting $T(x) = y_x$. It can be verified that the map T is **C**-linear.

Now clearly $(I + T)\mathscr{P}_W \cap [\mathscr{S}_\sigma(W) \cap \mathscr{R}^{m\times 1}(\sigma)] = (0)$ and

$$(I+T)\mathscr{P}_W \dotplus \left[\mathscr{S}_\sigma(W) \cap \mathscr{R}^{m\times 1}(\sigma)\right] \subset \mathscr{S}_\sigma(W).$$

Also, for every $h \in \mathscr{S}_\sigma(W)$ there exists $f \in \mathscr{P}_W$ such that $h - f \in \mathscr{R}^{m\times 1}(\sigma)$ and consequently $h - (I + T)f \in \mathscr{S}_\sigma(W) \cap \mathscr{R}^{m\times 1}(\sigma)$. Thus,

$$\mathscr{S}_\sigma(W) \subset (I+T)\mathscr{P}_W \dotplus \left[\mathscr{S}_\sigma(W) \cap \mathscr{R}^{m\times 1}(\sigma)\right]. \qquad \blacksquare$$

The map T occurring in Theorem 1.1. is called the *null-pole coupling operator*. It is unique once we fix $\mathscr{N}$. By (1.5), $\mathscr{S}_\pi$, $\mathscr{S}_\zeta$, $W^{\circ l}$, and T determine $\mathscr{S}_\sigma(W)$. It follows from Corollary 5.3 below that if a pole module $\mathscr{S}_\pi$, a zero module $\mathscr{S}_\zeta$, and $W^{\circ l}$ are given, there may exist different maps T_i such that the spaces defined by (1.5) are (different) null-pole subspaces of some rational matrix functions W_i.

In order to get a better insight into the nature of the null-pole coupling operator T, let $\Omega = W\mathscr{R}^{n\times 1}$. Suppose σ is a proper subset of $\mathbf{C}_\infty$, and choose Λ such that $\mathscr{R}^{m\times 1} = \Omega \oplus_\sigma \Lambda$. Let $\mathscr{N}_d$ be a **C**-linear subspace of Ω such that

$$\Omega \cap \mathscr{R}^{m\times 1}(\sigma) = \mathscr{N}_d \dotplus \left[\mathscr{S}_\sigma(W) \cap \mathscr{R}^{m\times 1}(\sigma)\right].$$

Let $\mathscr{N}_c = \Lambda \cap \mathscr{R}^{m\times 1}(\sigma)$. Since any function $h \in \mathscr{R}^{m\times 1}(\sigma)$ has a decomposition $h_\Lambda + h_\Omega$, where $h_\Lambda \in \Lambda \cap \mathscr{R}^{m\times 1}(\sigma)$ and $h_\Omega \in \Omega \cap \mathscr{R}^{m\times 1}(\sigma)$,

$$\mathscr{R}^{m\times 1}(\sigma) = \mathscr{N}_c \dotplus \mathscr{N}_d \dotplus \left[\mathscr{S}_\sigma(W) \cap \mathscr{R}^{m\times 1}(\sigma)\right].$$

Let $P_d : \mathscr{N} \to \mathscr{N}_d$ be the projection along $\mathscr{N}_c$, and let $P_c : \mathscr{N} \to \mathscr{N}_c$ be the projection along $\mathscr{N}_d$. Let $T_d = P_d T$ and let $T_c = P_c T$. Then $T = T_c + T_d$, and the decomposition (1.5) becomes

$$\mathscr{S}_\sigma(W) = (I + T_c + T_d)\mathscr{P}_W \dotplus \left[\mathscr{S}_\sigma(W) \cap \mathscr{R}^{m\times 1}(\sigma)\right]. \qquad (1.6)$$

If $W^{\circ l} = (0)$, then $\mathcal{N}_c = (0)$ and $T = T_d$ is the operator known from the regular case.

2. LEFT SPECTRAL TRIPLES

We want to describe a left null-pole subspace of a function $W \in \mathcal{R}^{m \times n}$ in an analytic way. To achieve this, we need a coordinate-wise description of the modules introduced in the previous section. We begin with the pole module $\mathcal{S}_\pi$ or, more precisely, with the module $\mathcal{P}_W$.

In [12], the definition of a pole pair of a regular rational matrix function over a subset of **C** has been extended to the nonregular case. An observable pair of matrices (C_π, A_π) of sizes $k_\pi \times n_\pi$ and $n_\pi \times n_\pi$ with $\sigma(A_\pi) \subset \sigma$ is a right pole pair over σ for a function $W \in \mathcal{R}^{m \times n}$ if

(i) for every $x \in \mathbf{C}^{n_\pi \times 1}$ there exists a function $\phi \in \mathcal{S}_\sigma(W)$ such that

$$C_\pi(z - A_\pi)^{-1}x - \phi(z) \in \mathcal{R}^{m \times 1}(\sigma), \tag{2.1}$$

(ii) for every function $\phi \in \mathcal{S}_\sigma(W)$ there exists a vector $x \in \mathbf{C}^{n_\pi \times 1}$ such that (2.1) holds.

It is clear from properties (i) and (ii) of a right pole pair for W over σ that the module $\mathcal{P}_W$ can be characterized as follows.

Proposition 2.1. *Let (C_π, A_π) be a right pole pair over σ for a function $W \in \mathcal{R}^{m \times n}$. Then*

$$\mathcal{P}_W = \left\{ C_\pi(z - A_\pi)^{-1}x : x \in \mathbf{C}^{n_\pi \times 1} \right\}, \tag{2.2}$$

where n_π is the number of columns of A_π.

Equation (2.2) shows that a right pole pair for W over σ determines $\mathcal{P}_W$ completely. On the other hand, the module $\mathcal{P}_W$ determines a right pole pair for W over σ up to right similarity, where pairs (E, F) and (G, H) are right similar if $G = ES$ and $H = S^{-1}FS$ for some nonsingular matrix S (see Proposition 1.3 in [12]). We note that the pairs (C_π, A_π) are well known in systems theory: if the function W is analytic in $\mathbf{C}_\infty \setminus \sigma$, then there exist matrices B and D such that

$$W(z) = D + C_\pi(z - A_\pi)^{-1}B,$$

i.e., $W(z)$ is the transfer function of a linear system having C_π as output map and A_π as state operator.

We describe the left annihilator of W by means of a triple of matrices $\tau_\kappa = (A_\kappa, B_\kappa, D_\kappa)$, a left kernel triple for W. In this description, the matrix A_κ is a nilpotent matrix in the Jordan form and the pair (A_κ, B_κ) is controllable. If D_κ has n rows and $A_\kappa = \operatorname{diag}(J_1, J_2, \dots, J_\eta)$, where J_i is the $k_i \times k_i$ Jordan block, let $C_\kappa = [E_1 \;\; E_2 \;\; \cdots \;\; E_\eta]$, where E_i is an $n \times k_i$ matrix with 1 at the $(i,1)$th position and zeros elsewhere. Then $(A_\kappa, B_\kappa, C_\kappa, D_\kappa)$ is a minimal realization of the associated matrix polynomial P_κ of τ_κ, that is,

$$P_\kappa(z) = D_\kappa + C_\kappa(z^{-1} - A_\kappa)^{-1} B_\kappa.$$

The triple τ_κ is such that the rows of the associated matrix polynomial P_κ span $W^{\circ l}$, they are orthogonal at infinity, and P_κ has no zeros in **C**. Any matrix polynomial with these three properties is called a left kernel polynomial for W. We note that the sizes of the Jordan blocks in A_κ coincide with the left Forney indices of W.

We define now an $\mathscr{R}(\sigma)$-module which is isomorphic to the left zero module $\mathscr{S}_\zeta$ of a function $W \in \mathscr{R}^{m \times n}$. By partial-fraction expansion, the **C**-vector space $\mathscr{R}^{1 \times m}$ has a direct-sum decomposition

$$\mathscr{R}^{1 \times m} = \mathscr{R}^{1 \times m}(\sigma) \dotplus \mathscr{R}_\circ^{1 \times m}(\sigma^c),$$

where $\mathscr{R}^{1 \times m}$ contains functions analytic on σ and $\mathscr{R}_\circ^{1 \times m}(\sigma^c)$ contains functions which are analytic on σ^c and vanish at infinity. Let P_{σ^c} be the projection of $\mathscr{R}^{1 \times m}$ onto $\mathscr{R}_\circ^{1 \times m}(\sigma^c)$ along $\mathscr{R}^{1 \times m}(\sigma)$. Then $(\mathscr{R}_\circ^{1 \times m}(\sigma^c), \cdot)$ is an $\mathscr{R}(\sigma)$-module with multiplication "$\cdot$" defined by

$$r \cdot h = P_{\sigma^c}(rh)$$

for all $r \in \mathscr{R}(\sigma)$ and $h \in \mathscr{R}_\circ^{1 \times m}(\sigma^c)$. Let EDF be a Smith-McMillan factorization of W, so that E and F are unimodular matrix polynomials and D is the Smith-McMillan form of W. Let k be number of nonzero entries of D (that is, the normal rank of W). Let d_i be the ith diagonal entry of D, let E_i be the ith column of E, and let $\mathscr{E}_i$ be the ith row of E^{-1} $(i = 1, 2, \dots, k)$. Define an $\mathscr{R}(\sigma)$-module homomorphism $\phi : W\mathscr{R}^{n \times 1} \cap \mathscr{R}^{m \times 1}(\sigma) \to (\mathscr{R}_\circ^{1 \times m}(\sigma^c), \cdot)$ by $\phi(E_i) = P_{\sigma^c}((1/d_i)\mathscr{E}_i)$, and let $\mathscr{Z}_W = \operatorname{Im} \phi$. Then $h = \sum_{i=1}^{k} r_i E_i$ is mapped under ϕ into 0 if and only if $r_i / d_i \in \mathscr{R}(\sigma)$ for $i = 1, 2, \dots, k$, which happens if and only if $h \in W\mathscr{R}^{n \times 1}(\sigma) \cap \mathscr{R}^{m \times 1}(\sigma)$. It follows that $\operatorname{Ker} \phi = \mathscr{S}_\sigma(W) \cap \mathscr{R}^{m \times 1}(\sigma)$ and hence $\mathscr{Z}_W \approx \mathscr{S}_\zeta$.

We describe $\mathcal{Y}_W$ in terms of a pair of matrices (A_ζ, B_ζ), called a left null pair for W. Let Λ be the subspace of $\mathcal{R}^{1\times m}$ spanned (over $\mathcal{R}$) by the first k rows of E^{-1}, viz. $\mathcal{E}_1, \mathcal{E}_2, \dots, \mathcal{E}_k$. Then (cf. Proposition 3.8, the definition of a left null pair, and Proposition 3.18 in [12]) there exists a pair of matrices (A_ζ, B_ζ), of sizes $n_\zeta \times n_\zeta$ and $n_\zeta \times m_\zeta$, respectively, such that

(i) if $x \in \mathbf{C}^{1\times n_\zeta}$, then for some $h \in \Lambda$ one has $hW \in \mathcal{R}^{1\times n}(\sigma)$ and

$$x(z - A_\zeta)^{-1}B_\zeta - h(z) \in \mathcal{R}^{1\times m}(\sigma); \tag{2.3}$$

(ii) if $h \in \Lambda$ and $hW \in \mathcal{R}^{1\times n}(\sigma)$, then (2.3) holds for some $x \in \mathbf{C}^{1\times n_\zeta}$.

The pair (A_ζ, B_ζ) is one way to measure the discrete zero structure of W on the left side. Indeed, if $h \in \Lambda$ and $hW \in \mathcal{R}^{1\times m}(\sigma)$, then the poles of h in σ (if any) are canceled by the zeros of W.

The pairs (A_ζ, B_ζ) occur in systems theory. More specifically, if the function W is invertible and all zeros of W are contained in σ, then there exist matrices C and D such that

$$W^{-1}(z) = D + C(z - A_\zeta)^{-1}B_\zeta,$$

that is, $W^{-1}(z)$ is the transfer function of a linear system having A_ζ, as state operator and B_ζ as input map.

The module $\mathcal{Y}_W$ can be analytically described as follows.

PROPOSITION 2.2. *Let* W, Λ, $\mathcal{Y}_W$, *and* (A_ζ, B_ζ) *be as above. Then*

$$\mathcal{Y}_W = \left\{x(z - A_\zeta)^{-1}B_\zeta : x \in \mathbf{C}^{1\times n_\zeta}\right\}. \tag{2.4}$$

Proof. Let $x \in \mathbf{C}^{1\times n_\zeta}$. We show

$$x(z - A_\zeta)^{-1}B_\zeta \in \mathcal{Y}_W. \tag{2.5}$$

By Property (i) of (A_ζ, B_ζ), we have $x(z - A_\zeta)^{-1}B_\zeta = P_{\sigma'}(h(z))$ for some $h = \sum_{i=1}^{k} r_i \mathcal{E}_i$ such that $hW \in \mathcal{R}^{1\times n}(\sigma)$. So $r_i d_i \in \mathcal{R}(\sigma)$ $(i = 1, 2, \dots, k)$, and hence h is contained in the image of ϕ.

Conversely, suppose $u \in \mathcal{Y}_W$. Then $u = P_{\sigma'}(h)$, where $h = \sum_{i=1}^{k} r_i \mathcal{E}_i$ for some $r_1, r_2, \dots, r_k$ such that $\sum_{i=1}^{k} r_i d_i E_i \in \mathcal{R}^{m\times 1}(\sigma)$. So $hW \in \mathcal{R}^{1\times n}(\sigma)$.

Hence, by property (ii) of the pair (A_ζ, B_ζ), we have $u \in \{x(z - A_\zeta)^{-1}B_\zeta : x \in \mathbf{C}^{1\times n_\zeta}\}$. ■

Let $\mathcal{S}_\pi$ be the pole module over $\sigma \subset \mathbf{C}$ of a function $W \in \mathcal{R}^{m\times n}$. Then, as can be easily seen from the definition, there is exactly one module $\mathcal{P}_W$ associated with W as defined in Section 1. In the language of pole pairs, this can be stated by saying that a right pole pair for W over σ is unique up to right similarity (cf. Proposition 2.1 above). The module $\mathcal{P}_W$ determines the *right pole structure* of W over σ. The definition of the module $\mathcal{Z}_W$ resembles the definition of the module $\mathcal{P}_W$. In contrast to $\mathcal{P}_W$, the module $\mathcal{Z}_W$ need not be uniquely determined by W, although it is isomorphic to the zero module $\mathcal{S}_\zeta$, which is uniquely determined by W. This is connected with the nonuniqueness of the first factor in a Smith-McMillan factorization EDF of W and the consequent nonuniqueness of the space Λ in the definition of a left null pair. In the language of left null pairs, this can be expressed by saying that if $W^{\circ l} \neq (0)$ then left null pairs for W over σ need not be left similar, where pairs (A, B) and (G, H) are left similar if $G = SAS^{-1}$ and $H = SB$ for some nonsingular matrix S. If the left annihilator of W is trivial, $\Lambda = \mathcal{R}^{1\times m}$ is unique and then the module $\mathcal{Z}_W$ is unique. For an analysis of the amount of nonuniqueness of a left null pair, see [1].

Suppose $W^{\circ l} \neq (0)$, and let $(\tilde{A}_\zeta, \tilde{B}_\zeta)$ be an arbitrary left null pair for W over σ, as defined in [12]. Then it follows from Lemma 3.16 in [12] that there exists a Smith-McMillan factorization $\tilde{E}D\tilde{F}$ of W such that if we replace E with $\tilde{E}$ in the definition of $\mathcal{Z}_W$ above, we shall obtain the module

$$\tilde{\mathcal{Z}}_W = \left\{x\left(z - \tilde{A}_\zeta\right)^{-1}\tilde{B}_\zeta : x \in \mathbf{C}^{1\times n_\zeta}\right\}. \tag{2.4'}$$

Modules $\mathcal{Z}_W$ and $\tilde{\mathcal{Z}}_W$ are equal if and only if the pairs (A_ζ, B_ζ) and $(\tilde{A}_\zeta, \tilde{B}_\zeta)$ are left similar. It follows that we can identify modules $\mathcal{Z}_W$ with left null pairs (A_ζ, B_ζ) for W over σ or, more precisely, with equivalence classes of left-similar left null pairs for W over σ.

The next proposition shows that a left null pair τ_ζ for W over $\sigma \subset \mathbf{C}$ and a left kernel triple τ_κ determine the free module $\mathcal{S}_\sigma(W) \cap \mathcal{R}^{m\times 1}(\sigma)$. Consequently, τ_ζ and τ_κ determine the zero module $\mathcal{S}_\zeta$ of W over σ.

PROPOSITION 2.3. *Let (A_ζ, B_ζ) be a left null pair for a function $W \in R^{m\times n}$ over a subset σ of* $\mathbf{C}$, *and let $\mathcal{Z}_W$ be as in* (2.4). *Then*

$$\mathcal{S}_\sigma(W) \cap \mathcal{R}^{m\times 1}(\sigma) = \left\{h \in W\mathcal{R}^{n\times 1} \cap \mathcal{R}^{m\times 1}(\sigma) : gh \in \mathcal{R}(\sigma)\ \forall g \in \mathcal{Z}_W\right\}. \tag{2.6}$$

Proof. If $h \in \mathscr{S}_\sigma(W) \cap \mathscr{R}^{m\times 1}(\sigma)$, then, by Proposition 3.18 in [12], $(z - A_\zeta)^{-1}B_\zeta h(z)$ is analytic on σ. Hence, in view of Proposition 2.2, $gh \in \mathscr{R}(\sigma)$ for every $g \in \mathscr{G}_W$ and h is a member of the set on the right-hand side of (2.6).

Suppose h is a member of the set on the right-hand side of (2.6). Then, if $d_i, E_i, \mathscr{E}_i$ $(i = 1,2,\dots,k)$ are as above, $h = \sum_{i=1}^{k} r_i d_i E_i$ for some $r_1, r_2, \dots, r_k \in \mathscr{R}$. Since $h \in \mathscr{R}^{m\times 1}(\sigma)$ and the columns of E_i are orthogonal on σ, we have $r_i d_i \in \mathscr{R}(\sigma)$ for $i = 1,2,\dots,k$ (cf. Corollary 2.4 in [12]). So r_i is analytic at every point $\lambda \in \sigma$ where d_i does not vanish. Suppose $\lambda \in \sigma$ is a zero of W of geometric multiplicity μ, and let $d_{k-\mu}, d_{k-\mu+1}, \dots, d_k$ be the nonzero diagonal entries of the Smith-McMillan form D of W which vanish at λ to the orders $l_{k-\mu}, l_{k-\mu+1}, \dots, l_k$, respectively. Then $(z - \lambda)^{-l_{k-j}}\mathscr{E}_{k-j}(z)W(z)$ is analytic on σ for $j = 0,1,\dots,\mu$. By property (ii) of (A_ζ, B_ζ), $\mathscr{G}_W$ contains functions $\phi_{k-\mu}, \phi_{k-\mu+1}, \dots, \phi_k$ such that ϕ_{k-j} is the singular part of the Laurent expansion at λ of $(z - \lambda)^{-l_{k-j}}\mathscr{E}_{k-j}(z)$. Then $\phi_{k-j}E_i$ is analytic at λ whenever $i \neq k - j$. Since, by hypothesis, $\phi_{k-j}h \in \mathscr{R}(\sigma)$, r_{k-j} is analytic at λ for $j = 0,1,\dots,\mu$. It follows that $h \in \mathscr{S}_\sigma(W)$. ■

Proposition 2.3 can be expressed analytically as follows.

LEMMA 2.4. *Let $\sigma \subset \mathbf{C}$, and let (A_ζ, B_ζ) be a left null pair over σ for a function $W \in \mathscr{R}^{m\times n}$. Let τ_κ be a left kernel triple for W, and let Ω be the subspace of $\mathscr{R}^{m\times 1}$ annihilated by the matrix polynomial associated with τ_κ. Then*

$$\mathscr{S}_\sigma(W) \cap \mathscr{R}^{m\times 1}(\sigma) = \Big\{ h \in \Omega \cap \mathscr{R}^{m\times 1}(\sigma) : \sum_{z_0 \in \sigma} \operatorname{Res}_{z=z_0}(z - A_\zeta)^{-1}B_\zeta h(z) = 0 \Big\}.$$

Proof. By Proposition 3.18 in [12], $(z - A_\zeta)^{-1}B_\zeta h(z)$ is analytic on σ whenever $h \in \mathscr{S}_\sigma(W) \cap \mathscr{R}^{m\times 1}(\sigma)$. So

$$\mathscr{S}_\sigma(W) \cap \mathscr{R}^{m\times 1}(\sigma) \subset \Big\{ h \in \Omega \cap \mathscr{R}^{m\times 1}(\sigma) : \sum_{z_0 \in \sigma} \operatorname{Res}_{z=z_0}(z - A_\zeta)^{-1}B_\zeta h(z) = 0 \Big\}. \quad (2.7)$$

Let h be a member of the set on the right-hand side of the inclusion (2.7), and let $\mathscr{P}_W$ be the module corresponding to (A_ζ, B_ζ) as in Proposition 2.2. We may assume that A_ζ is in Jordan form. After considering the rows of the matrix function $(z - A_\zeta)^{-1}B_\zeta$, it is easy to see that $gh \in \mathscr{R}(\sigma)$ for every $g \in \mathscr{P}_W$. Hence, by Proposition 2.3, $h \in \mathscr{S}_\sigma(W)$ and the inclusion (2.7) can be reversed. ■

Let $\sigma \subset \mathbf{C}$ and let $W \in \mathscr{R}^{m \times n}$. Let $\mathscr{N}$ be a complement of $\mathscr{S}_\sigma(W) \cap \mathscr{R}^{m \times 1}(\sigma)$ in $\mathscr{R}^{m \times 1}(\sigma)$, and let $T : \mathscr{P}_W \to \mathscr{N}$ be the coupling operator such that the equality (1.5) holds. If (C_π, A_π) is a right pole pair and (A_ζ, B_ζ) is a left null pair for W over σ, let

$$\Gamma = \sum_{z_0 \in \sigma} \operatorname{Res}_{z = z_0}(z - A_\zeta)^{-1} B_\zeta \begin{bmatrix} h_1(z) & h_2(z) & \cdots & h_{n_\pi}(z) \end{bmatrix}, \quad (2.8)$$

where $h_i = T(C_\pi(z - A_\pi)^{-1}e_i)$ for $i = 1, 2, \ldots, n_\pi$ (e_i denotes here the standard vector with 1 at the ith position and zeros elsewhere). The $n_\zeta \times n_\pi$ matrix Γ is called the *null-pole coupling matrix* associated with a right pole pair (C_π, A_π) and a left null pair (A_ζ, B_ζ) for W over σ. It is worthwhile to point out that Γ does not depend on the choice of $\mathscr{N}$. Indeed, if $\tilde{\mathscr{N}}$ is another complement of $\mathscr{S}_\sigma(W) \cap \mathscr{R}^{m \times 1}(\sigma)$ in $\mathscr{R}^{m \times 1}(\sigma)$ and $\tilde{T}$ is the associated coupling operator, then

$$T\big(C_\pi(z - A_\pi)^{-1}e_i\big) - \tilde{T}\big(C_\pi(z - A_\pi)^{-1}e_i\big) \in \mathscr{S}_\sigma(W) \cap \mathscr{R}^{m \times 1}(\sigma).$$

Hence, by Lemma 2.4, the right-hand side of (2.8) is equal to

$$\sum_{z_0 \in \sigma} \operatorname{Res}_{z = z_0}(z - A_\zeta)^{-1} B_\zeta \begin{bmatrix} \tilde{h}_1(z) & \tilde{h}_2(z) & \cdots & \tilde{h}_{n_\pi}(z) \end{bmatrix},$$

where $\tilde{h}_i = \tilde{T}_i(C_\pi(z - A_\pi)^{-1}e_i)$ for $i = 1, 2, \ldots, n_\pi$.

We show now that the null-pole coupling matrix can be defined in a more compact way.

LEMMA 2.5. *Let (A_ζ, B_ζ) be a left null pair of a function $W \in R^{m \times n}$ over a subset σ of* $\mathbf{C}$, *and let Ξ be a subspace of $\mathscr{R}^{1 \times m}$ such that (A_ζ, B_ζ) is left-similar to a pair constructed from Taylor coefficients of functions contained in Ξ. Then, for every $h \in \mathscr{R}^{m \times 1}(\sigma)$ which is annihilated by Ξ,*

$$\sum_{z_0 \in \sigma} \operatorname{Res}_{z = z_0}(z - A_\zeta)^{-1} B_\zeta h(z) = 0.$$

Proof. After multiplying both sides of the equality in question by a constant invertible matrix, we may assume that A_ζ is in Jordan form and the rows of B_ζ contain Taylor coefficients at the zeros of W in σ of left null functions of W which are contained in Ξ (see the definition of a left null pair in [4]). We may also assume that $\sigma(A_\zeta)$ contains a single point λ. Then each row of $(z-A_\zeta)^{-1}B_\zeta$ contains the singular part at λ of a function $(z-\lambda)^{-i}\phi_j(z)$, where i is a positive integer and $\phi_j \in \Xi$. So, since $\phi_j h = 0$, each row of $(z-A_\zeta)^{-1}B_\zeta$ is a sum of a function which annihilates h and a function which is analytic at λ. Since h is analytic at λ, the lemma follows. ■

If Λ is a set in $\mathscr{R}^{1\times m}$ (in $\mathscr{R}^{m\times 1}$), we use the symbol Λ° to denote the subspace of $\mathscr{R}^{m\times 1}$ (of $\mathscr{R}^{1\times m}$) which annihilates Λ. Also, if τ_ζ is a left null pair for a function $W \in \mathscr{R}^{m\times n}$ over $\sigma \subset \mathbf{C}$, and Λ is such that $\mathscr{R}^{1\times m} = W^{\circ l} \oplus_\sigma \Lambda$ and τ_ζ is left similar to a pair constructed from Taylor coefficients of functions in Λ, then we shall call Λ a subspace *associated* with τ_ζ.

LEMMA 2.6. *Let $\tau_\zeta = (A_\zeta, B_\zeta)$ with $A_\zeta \in \mathbf{C}^{n_\zeta \times n_\zeta}$ be a left null pair for a function $W \in \mathscr{R}^{m\times n}$ over $\sigma \subset \mathbf{C}$, and let Λ be an associated subspace. Define $\Phi : \mathscr{R}^{m\times 1}(\sigma) \to \mathbf{C}^{n_\zeta \times 1}$ by*

$$\Phi(h) = \sum_{z_0 \in \sigma} \operatorname{Res}_{z=z_0} (z-A_\zeta)^{-1} B_\zeta h(z).$$

Then

$$\operatorname{Ker}\Phi = \left[\Lambda^\circ \cap \mathscr{R}^{m\times 1}(\sigma)\right] \dot{+} \left[\mathscr{S}_\sigma(W) \cap \mathscr{R}^{m\times 1}(\sigma)\right].$$

Proof. By Theorem 2.8 in [12], $\mathscr{R}^{m\times 1} = \Lambda^\circ \oplus_\sigma (W\mathscr{R}^{n\times 1})$. Hence, by (0.2),

$$\mathscr{R}^{m\times 1}(\sigma) = \left[\Lambda^\circ \cap \mathscr{R}^{m\times 1}(\sigma)\right] \dot{+} \left[W\mathscr{R}^{n\times 1} \cap \mathscr{R}^{m\times 1}(\sigma)\right].$$

By Lemma 2.5, $\Lambda^\circ \cap \mathscr{R}^{m\times 1}(\sigma) \subset \operatorname{Ker}\Phi$. So the result follows by Lemma 2.4. ■

Let W, σ, τ_ζ, and Λ be as in Lemma 2.6. Let $\mathscr{N}_c = \Lambda^\circ \cap \mathscr{R}^{m\times 1}(\sigma)$, and choose $\mathscr{N}_d$ such that

$$W\mathscr{R}^{n\times 1} \cap \mathscr{R}^{m\times 1}(\sigma) = \mathscr{N}_d \dot{+} \left[\mathscr{S}_\sigma(W) \cap \mathscr{R}^{m\times 1}(\sigma)\right].$$

We saw in Section 1 that if $\mathscr{N} = \mathscr{N}_c \dot{+} \mathscr{N}_d$, then the associated coupling operator T splits into two parts, $T_c : \mathscr{P}_W \to \mathscr{N}_c$ and $T_d : \mathscr{P}_W \to \mathscr{N}_d$. In view of Lemma 2.6, the null-pole coupling matrix is equal to

$$\Gamma = \sum_{z_0 \in \sigma} \operatorname{Res}_{z=z_0} (z - A_\zeta)^{-1} B_\zeta \begin{bmatrix} h_1(z) & h_2(z) & \cdots & h_{n_\pi}(z) \end{bmatrix}, \tag{2.9}$$

where $h_i = T_d(C_\pi(z - A_\pi)^{-1} e_i) \in \mathscr{N}_d$ $(i = 1, 2, \dots, n_\pi)$.

The ordered triple

$$\tau_s = \big((C_\pi, A_\pi), (A_\zeta, B_\zeta), \Gamma\big) \tag{2.10}$$

is called a *left spectral triple* for W over σ, or a *left σ-spectral triple* for W. Plainly, different right pole and left null pairs for W over σ give rise to different left σ-spectral triples for W. In particular, if we take $(\tilde{C}_\pi, \tilde{A}_\pi) = (C_\pi S_\pi, S_\pi^{-1} A_\pi S_\pi)$ and $(\tilde{A}_\zeta, \tilde{B}_\zeta) = (S_\zeta A_\zeta S_\zeta^{-1}, S_\zeta B_\zeta)$ instead of (C_π, A_π) and (A_ζ, B_ζ), where S_π and S_ζ are any nonsingular matrices of appropriate sizes, we shall obtain the coupling matrix $\tilde{\Gamma} = S_\zeta \Gamma S_\pi$. Thus, for any nonsingular matrices S_π and S_ζ of appropriate sizes, the triple

$$\begin{aligned} \tilde{\tau}_s &= \big((\tilde{C}_\pi, \tilde{A}_\pi), (\tilde{A}_\zeta, \tilde{B}_\zeta), \tilde{\Gamma}\big) \\ &= \big((C_\pi S_\pi, S_\pi^{-1} A_\pi S_\pi), (S_\zeta A_\zeta S_\zeta^{-1}, S_\zeta B_\zeta), S_\zeta \Gamma S_\pi\big) \end{aligned}$$

is a left σ-spectral triple for W whenever τ_s as in (2.10) is. The triples τ_s and $\tilde{\tau}_s$ are said to be *similar*. If the left annihilator of W is trivial, all left σ-spectral triples for W are similar. If $W^{\circ l} \neq (0)$, W may have left null pairs over σ which are not left similar. Then W will have left spectral triples which are not similar.

A left σ-spectral triple and a left kernel triple for a function W determine completely the left null-pole subspace for W over σ.

THEOREM 2.7. *Let $\sigma \subset \mathbf{C}$, and let $((C_\pi, A_\pi), (A_\zeta, B_\zeta), \Gamma)$ be a left spectral triple over σ for a function $W \in \mathscr{R}^{m \times n}$. Let τ_κ be a left kernel triple for W, and let Ω be the subspace annihilated by the matrix polynomial associated with τ_κ. Then*

$$\mathscr{S}_0(W) = \Omega \cap \Big\{ C_\pi (z - A_\pi)^{-1} x + h(z) : x \in \mathbf{C}^{n_\pi \times 1},\ h \in \mathscr{R}^{m \times 1}(\sigma), \\ \textit{and} \sum_{z_0 \in \sigma} \operatorname{Res}_{z=z_0} (z - A_\zeta)^{-1} B_\zeta h(z) = \Gamma x \Big\}. \tag{2.11}$$

Proof. Let $f \in \mathscr{S}_\sigma(W)$. By Proposition 2.1, the projection of f onto $\mathscr{R}_0^{m\times 1}(\sigma^c)$ along $\mathscr{R}^{m\times 1}(\sigma)$ is

$$f_{\sigma^c}(z) = C_\pi(z - A_\pi)^{-1}x$$

for some $x \in \mathbf{C}^{n_\pi \times 1}$. Then $f = (I+T)f_{\sigma^c} + f_\sigma$, where $f_\sigma \in \mathscr{S}_\sigma(W) \cap \mathscr{R}^{m\times 1}(\sigma)$ and T has the properties listed in Theorem 1.1. By the definition of the null-pole coupling matrix, $\tilde{f} = T(f_{\sigma^c})$ is a function such that $\tilde{f} \in \mathscr{R}^{m\times 1}(\sigma)$, $f_{\sigma^c} + \tilde{f} \in \mathscr{S}_\sigma(W)$, and

$$\sum_{z_0 \in \sigma} \operatorname{Res}_{z=z_0}(z - A_\zeta)^{-1}B_\zeta \tilde{f}(z) = \Gamma x. \tag{2.12}$$

Since $f \in \Omega$, in view of Lemma 2.4 f is a member of the right-hand side of the equality (2.11).

Suppose now f is a member of right-hand side of the equality (2.11). Then $f(z) = C_\pi(z - A_\pi)^{-1}x + h(z)$ for some $x \in \mathbf{C}^{n_\pi \times 1}$ and some $h \in \mathscr{R}^{m\times 1}(\sigma)$. So $f_{\sigma^c}(z) = C_\pi(z - A_\pi)^{-1}x$ is the projection of f onto $\mathscr{R}_0^{m\times 1}(\sigma^c)$ along $\mathscr{R}^{m\times 1}(\sigma)$. In view of Proposition 2.1, $f_{\sigma^c} \in \mathscr{P}_W$, where $\mathscr{P}_W$ is the module defined in Section 1. By Theorem 1.1 and the definition of Γ, there exists a function $\tilde{f} \in \mathscr{R}^{m\times 1}(\sigma)$ such that $f_{\sigma^c} + \tilde{f} \in \mathscr{S}_\sigma(W)$ and (2.12) holds. Hence $\tilde{h} = (f_{\sigma^c} + h) - (f_{\sigma^c} + \tilde{f}) \in \Omega \cap \mathscr{R}^{m\times 1}(\sigma)$ and, by linearity,

$$\sum_{z_0 \in \sigma} \operatorname{Res}_{z=z_0}(z - A_\zeta)^{-1}B_\zeta \tilde{h}(z) = 0.$$

By Lemma 2.4, $\tilde{h} \in \mathscr{S}_\sigma(W)$. So $f \in \mathscr{S}_\sigma(W)$. ■

The characterization of $\mathscr{S}_\sigma(W)$ in Theorem 2.7 utilizes a left σ-spectral triple for W. It is a coordinatewise counterpart of the basis-free description given in Theorem 1.1.

3. PROPERTIES OF NULL-POLE COUPLING MATRICES

Basic properties of right pole and left null pairs of a function $W \in \mathscr{R}^{m\times n}$ over a subset σ of $\mathbf{C}$ have been indicated in [12]. Now we show basic properties of a coupling matrix Γ. We derive first the Sylvester equation which Γ always satisfies.

THEOREM 3.1. *If $\tau_s = ((C_\pi, A_\pi),(A_\zeta, B_\zeta),\Gamma)$ is a left spectral triple for a function $W \in \mathscr{R}^{m\times n}$ over a subset σ of $\mathbf{C}$, then*

$$\Gamma A_\pi - A_\zeta \Gamma = B_\zeta C_\pi . \tag{3.1}$$

Proof. For notational convenience, we assume $0 \in \sigma$. Let Λ be a subspace associated with (A_ζ, B_ζ), let $\mathscr{N}_c = \Lambda^\circ \cap \mathscr{R}^{m\times 1}(\sigma)$, and choose $\mathscr{N}_d$ such that $W\mathscr{R}^{n\times 1} \cap \mathscr{R}^{m\times 1}(\sigma) = \mathscr{N}_d \dot{+} [\mathscr{S}_\sigma(W) \cap \mathscr{R}^{m\times 1}(\sigma)]$. Let $T_c : \mathscr{P}_W \to \mathscr{N}_c$ and $T_d : \mathscr{P}_W \to \mathscr{N}_d$ be such that (1.6) holds. Then, if $f \in \mathscr{P}_W$, we have $M_z(I + T_c + T_d)f \in \mathscr{S}_\sigma(W)$, where M_z is the operator of multiplication by z. So

$$M_z f + M_z T_c f + M_z T_d f = g + T_c g + T_d g + h \tag{3.2}$$

for some $g \in \mathscr{P}_W$ and $h \in \mathscr{S}_\sigma(W) \cap \mathscr{R}^{m\times 1}(\sigma)$. Let $P_{\sigma^c} : \mathscr{R}^{m\times 1} \to \mathscr{R}_\circ^{m\times 1}(\sigma^c)$, $P_\sigma : \mathscr{R}^{m\times 1} \to \mathscr{R}^{m\times 1}(\sigma)$, and $P_{\mathscr{N}_d} : W\mathscr{R}^{n\times 1} \cap \mathscr{R}^{m\times 1}(\sigma) \to \mathscr{N}_d$ be the projections along $\mathscr{R}^{m\times 1}(\sigma)$, $\mathscr{R}_\circ^{m\times 1}(\sigma^c)$, and $\mathscr{N}_c \dot{+} [\mathscr{S}_\sigma(W) \cap \mathscr{R}^{m\times 1}(\sigma)]$, respectively. Then the equality (3.2) implies that

$$g = P_{\sigma^c} M_z f$$

and

$$T_d g = P_{\mathscr{N}_d}(P_\sigma M_z f + M_z T_d f).$$

So

$$T_d P_{\sigma^c} M_z f - P_{\mathscr{N}_d} M_z T_d f = P_{\mathscr{N}_d} P_\sigma M_z f$$

for every $f \in \mathscr{P}_W$. Define $\mathscr{C} : \mathscr{R}^{m\times 1}(\sigma) \to \mathbf{C}^{n_\zeta\times 1}$ and $\mathscr{O} : \mathbf{C}^{n_\zeta\times 1} \to \mathscr{P}_W$ by $\mathscr{C}(h) = \sum_{z_0\in\sigma} \operatorname{Res}_{z=z_0}(z - A_\zeta)^{-1} B_\zeta h(z)$ and $\mathscr{O}(x) = C_\pi(z - A_\pi)^{-1}x$. Then $\mathscr{C}$ and $\mathscr{O}$ are linear operators and

$$\mathscr{C} T_d P_{\sigma^c} M_z \mathscr{O} x - \mathscr{C} P_{\mathscr{N}_d} M_z T_d \mathscr{O} x = \mathscr{C} P_{\mathscr{N}_d} P_\sigma M_z \mathscr{O} x \tag{3.3}$$

for every $x \in \mathbf{C}^{n_\pi\times 1}$. Now $P_{\sigma^c} M_z \mathscr{O} = \mathscr{O} A_\pi$ and $P_\sigma M_z \mathscr{O} = C_\pi$. Also, by Lemma 2.6, $\mathscr{C} P_{\mathscr{N}_d} M_z T_d = \mathscr{C} M_z T_d = \mathscr{A}_\zeta \mathscr{C}$ and $\mathscr{C} P_{\mathscr{N}_d} C_\pi x = \mathscr{C} C_\pi = B_\zeta C_\pi$. Thus, (3.3) implies

$$\mathscr{C} T_d \mathscr{O} A_\pi x - A_\zeta \mathscr{C} T_d \mathscr{O} x = B_\zeta C_\pi x$$

for every $x \in \mathbf{C}^{n_\pi\times 1}$. Since, by (2.9), $\mathscr{C} T_d \mathscr{O} = \Gamma$, the equality (3.1) follows. ■

Below, we shall need the following lemma.

LEMMA 3.2. *Let* $W_1 \in \mathscr{R}^{m\times n_1}$ *and* $W_2 \in R^{m\times n_2}$ *be functions with left spectral triples over a subset* σ *of* $\mathbf{C}$ *equal to* $((C_\pi, A_\pi),(A_\zeta, B_\zeta),\Gamma_1)$ *and* $((C_\pi, A_\pi),(A_\zeta, B_\zeta),\Gamma_2)$, *respectively. If* $W_1\mathscr{R}^{n_1\times 1}(\sigma)\subset W_2\mathscr{R}^{n_2\times 1}(\sigma)$, *then* $\Gamma_1 = \Gamma_2$.

Proof. We argue by contradiction. Suppose $\Gamma_1 \neq \Gamma_2$, and let x_0 be a constant vector such that $\Gamma_1 x_0 \neq \Gamma_2 x_0$. By Lemma 3.2.2 in [4], there exists a function $\phi \in \mathscr{R}^{m\times 1}(\sigma)$ such that

$$\sum_{z_0\in\sigma} \operatorname{Res}_{z=z_0}(z-A_\zeta)^{-1}B_\zeta\phi(z) = \Gamma_1 x_0.$$

We will utilize x_0 and ϕ to exhibit a function $\psi \in W_1\mathscr{R}^{n_1\times 1}\backslash W_2\mathscr{R}^{n_2\times 1}$.

Let P_1 be a left kernel polynomial of W_1, and let Λ_1 be the row span (over $\mathscr{R}$) of the matrix polynomial P_1. By Lemma 3.16 in [12], there exists a subspace Λ_2 of $\mathscr{R}^{1\times m}$, orthogonal to Λ_1 on σ, such that the pair (A_ζ, B_ζ) is left similar to a pair constructed from Taylor coefficients of functions in Λ_2. By Corollary 2.6 in [12], there exists a subspace Λ_3, orthogonal to $\Lambda_1+\Lambda_2$ on σ, such that

$$\mathscr{R}^{1\times m} = \Lambda_1 \oplus_\sigma \Lambda_2 \oplus_\sigma \Lambda_3,$$

that is, Λ_i is orthogonal on σ to $\Lambda_j + \Lambda_k$ $(i\neq j,\ i\neq k,\ j\neq k)$. Let

$$\Omega_1 = (\Lambda_2+\Lambda_3)^\circ, \qquad \Omega_2 = (\Lambda_1+\Lambda_3)^\circ, \qquad \Omega_3 = (\Lambda_1+\Lambda_2)^\circ,$$

where $(*)^\circ$ denotes the subspace of $\mathscr{R}^{m\times 1}$ which is annihilated by $*$. Then, by Theorem 2.8 in [12],

$$\mathscr{R}^{m\times 1} = \Omega_1 \oplus_\sigma \Omega_2 \oplus_\sigma \Omega_3.$$

Let $\phi_i \in \Omega_i$ $(i=1,2,3)$ be such that $\phi = \phi_1+\phi_2+\phi_3$. By Proposition 4.3 in [12], $\phi_i \in \mathscr{R}^{m\times 1}(\sigma)$ $(i=1,2,3)$. Let $w\in\Omega_1$ be such that $P_1[C_\pi(z-A_\pi)^{-1}x_0 + w] = 0$. Since $P_1C_\pi(z-A_\pi)^{-1}x_0 \in \mathscr{R}^{m\times 1}(\sigma)$ (see Proposition 4.1 in [12]), by Proposition 4.6 in [12] we have $w\in\mathscr{R}^{m\times 1}(\sigma)$. Hence

$w + \phi_2 \in \mathscr{R}^{m\times 1}(\sigma)$ and $\psi(z) = C_\pi(z - A_\pi)^{-1}x_0 + w(z) + \phi_2(z) \in \operatorname{Ker} P_1$. Also, by Lemma 2.5,

$$\begin{aligned}\sum_{z_0\in\sigma} \operatorname{Res}_{z=z_0}(z - A_\zeta)^{-1}B_\zeta[w(z) + \phi_2(z)] &\\ = \sum_{z_0\in\sigma} \operatorname{Res}_{z=z_0}(z - A_\zeta)^{-1}B_\zeta\phi(z) &\\ = \Gamma_1 x_0. &\end{aligned}$$

Therefore, by Theorem 2.7, $\psi \in W_1\mathscr{R}^{n_1\times 1}(\sigma)$.

It remains to show $\psi \notin W_2\mathscr{R}^{n_2\times 1}(\sigma)$. Let $\mathscr{R}_0^{m\times 1}(\sigma^c)$ denote the set of $m\times 1$ rational vector functions which are analytic on the complement of σ and vanish at infinity. Then

$$\mathscr{R}^{m\times 1} = \mathscr{R}^{m\times 1}(\sigma) \dotplus \mathscr{R}_0^{m\times 1}(\sigma^c),$$

where all vector spaces are over $\mathbf{C}$. Since the pair (C_π, A_π) is observable, by Lemma 3.2.2 in [4] the function $x \to C_\pi(z - A_\pi)^{-1}x$ is an injection $\mathbf{C}^{n_\pi\times 1} \to \mathscr{R}_0^{m\times 1}(\sigma^c)$. Therefore the function $w + \phi_2$ in the definition of ψ is unique and, by the choice of x_0,

$$\psi \notin \left\{C_\pi(z - A_\pi)^{-1}x + h(z) : x \text{ is a constant vector}, h \in \mathscr{R}^{m\times 1}(\sigma), \text{ and} \right.$$
$$\left. \sum_{z_0\in\sigma} \operatorname{Res}_{z=z_0}(z - A_\zeta)^{-1}B_\zeta h(z) = \Gamma_2 x\right\}.$$

So, by Theorem 2.7, $\psi \notin W_2\mathscr{R}^{n_2\times 1}(\sigma)$. ■

Lemma 3.2 implies the following uniqueness property of null-pole coupling matrices.

THEOREM 3.3. *Let* $W_1 \in \mathscr{R}^{m\times n_1}$ *and* $W_2 \in \mathscr{R}^{m\times n_2}$ *be functions with the same left kernel triple and with left spectral triples over a subset* σ *of* $\mathbf{C}$ *equal to* $((C_\pi, A_\pi), (A_\zeta, B_\zeta), \Gamma_1)$ *and* $((C_\pi, A_\pi), (A_\zeta, B_\zeta), \Gamma_2)$, *respectively. Then* $W_1\mathscr{R}^{n_1\times 1}(\sigma) = W_2\mathscr{R}^{n_2\times 1}(\sigma)$ *if and only if* $\Gamma_1 = \Gamma_2$.

Proof. If $\Gamma_1 = \Gamma_2$, then $W_1\mathscr{R}^{n_1\times 1}(\sigma) = W_2\mathscr{R}^{n_2\times 1}(\sigma)$ by Theorem 2.7. If $W_1\mathscr{R}^{n_1\times 1}(\sigma) = W_2\mathscr{R}^{n_2\times 1}(\sigma)$, then $\Gamma_1 = \Gamma_2$ by Lemma 3.2. ■

Theorem 3.3 has the following corollary.

COROLLARY 3.4. *Let τ_s be a left spectral triple of a function $W \in \mathscr{R}^{m\times n}$ over a subset σ of* **C**. *Let τ_κ be a left kernel triple of W. Then a rational matrix function H has the same left null-pole subspace over σ if and only if τ_s is a left σ-spectral triple and τ_κ is a left kernel triple of H.*

Let $\sigma \subset \mathbf{C}$, and let $W, H \in \mathscr{R}^{m\times n}$ be functions such that $W = HQ$ for some Q with $Q, Q^{-1} \in \mathscr{R}^{n\times n}(\sigma)$. Such functions are said to be right equivalent on σ. It is shown in [11] that W and H are right equivalent on σ if and only if $W = HQ_1$ and $H = WQ_2$ for some $Q_1, Q_2 \in \mathscr{R}^{n\times n}(\sigma)$, which happens if and only if $\mathscr{S}_\sigma(W) = \mathscr{S}_\sigma(H)$. Corollary 3.4 characterizes right equivalence on σ in terms of spectral data.

Let $W \in \mathscr{R}^{m\times n}$ be a function with the normal rank k, and let (C_π, A_π) and (A_ζ, B_ζ) be right pole and left null pairs for W over a subset σ of **C**. Choose a Smith-McMillan factorization EDF of W, and let $\tilde{D}$ be a function formed by the first k columns of D. Let Λ be a subspace associated with (A_ζ, B_ζ). Find a basis $\{v_1, v_2, \dots, v_\kappa\}$ for the subspace of $\mathscr{R}^{m\times 1}$ annihilated by Λ such that the function $[v_1 \;\; v_2 \;\; \cdots \;\; v_\kappa]$ has neither zeros nor poles in $\sigma(A_\pi) \cup \sigma(A_\zeta)$. Let

$$W_E = \begin{bmatrix} E\tilde{D} & v_1 & v_2 & \cdots & v_\kappa \end{bmatrix}. \tag{3.4}$$

Then W_E is a regular rational matrix function with (C_π, A_π) as a right pole pair and (A_ζ, B_ζ) as a left null pair over $\sigma(A_\pi) \cup \sigma(A_\zeta)$ (cf. Proposition 3.10 in [12]). It follows from Lemma 3.2 that if Γ is the null-pole coupling matrix associated with the right pole pair (C_π, A_π) and a left null pair (A_ζ, B_ζ) of W_E over σ, then $\tau_s = ((C_\pi, A_\pi), (A_\zeta, B_\zeta), \Gamma)$ is a left σ-spectral triple for W. Thus, a null-pole coupling matrix associated with a right pole pair (C_π, A_π) and a left null pair (A_ζ, B_ζ) for W over σ can be computed as follows:

1. Find W_E as in (3.4).
2. Find $\tilde{W}_E$ which is right equivalent to W_E on $\sigma(A_\pi) \cup \sigma(A_\zeta)$ and has value I at infinity.
3. Find a minimal realization (A, B, C, I) for $\tilde{W}_E$, and compute Γ as in [6], Chapter 4.

The computations involved may be, however, extensive.

4. $\mathscr{R}(\sigma)$-SUBMODULES OF $\mathscr{R}^{m\times 1}$

Let σ be a subset of $\mathbf{C}_\infty$, and let S be an $\mathscr{R}(\sigma)$-submodule of $\mathscr{R}^{m\times 1}$. We are interested in the question when there exists a rational matrix function with S as a left null-pole subspace. In order to understand the problem better, we need to introduce some terminology. We shall say that S has a *pole* at a point λ if $\lambda \in \sigma$, some function in S has a pole at λ, and, for some integer k, all functions in $(z-\lambda)^k S$ are analytic at λ. We shall say that S has an *essential singularity* at λ if for each integer k the set $(z-\lambda)^k S$ contains functions with a pole at λ.

If $\sigma \subset \mathbf{C}_\infty$, we continue to denote by P_{σ^c} the projection of $\mathscr{R}^{m\times 1}$ onto $\mathscr{R}_\circ^{m\times 1}(\sigma^c)$ along $\mathscr{R}^{m\times 1}(\sigma)$.

Proposition 4.1. *Let σ be a subset of* $\mathbf{C}$, *and let S be an $\mathscr{R}(\sigma)$-submodule of $\mathscr{R}^{m\times 1}$ with a finite number of poles and without essential singularities. Then there exists an observable pair of matrices (C_π, A_π), of sizes $m \times n_\pi$ and $n_\pi \times n_\pi$, such that $\sigma(A_\pi) \subset \sigma$ and*

$$P_{\sigma^c}(S) = \left\{ C_\pi (z - A_\pi)^{-1} x : x \in \mathbf{C}^{n_\pi \times 1} \right\}. \tag{4.1}$$

Proof. We adapt the construction of a canonical set of pole functions for a given rational matrix function from [3] to a module setting. Let $w_1, w_2, \dots, w_\nu$ be the poles of S. We can choose a function $\phi_1 \in S$ which has a pole at w_1 of maximal possible order. Inductively, suppose we have chosen functions $\phi_1, \phi_2, \dots, \phi_\eta \in S$ such that $[\phi_1(z)]_{z=w_1}, \dots, [\phi_\eta(z)]_{z=w_1}$ are linearly independent, where $[\phi(z)]_{z=\lambda}$ denotes the leading coefficient in the Laurent expansion of ϕ at λ. If $\operatorname{span}\{[\phi_1(z)]_{z=w_1}, [\phi_2(z)]_{z=w_1}, \dots, [\phi_i(z)]_{z=w_1}\}$ contains the leading coefficients in the Laurent expansions at w_1 of all functions in S which have a pole at w_1, we take $\eta = i$ and we stop. Otherwise, choose a function $\phi_{i+1} \in S$ such that $[\phi_{i+1}(z)]_{z=w_1} \notin \operatorname{span}\{[\phi_1(z)]_{z=w_1}, [\phi_2(z)]_{z=w_1}, \dots, [\phi_i(z)]_{z=w_1}\}$ and ϕ_{i+1} has a pole at w_1 of maximal possible order. Now since $\mathbf{C}^{m\times 1}$ is finite dimensional, the process must terminate after finitely many steps. Let $\delta(\phi_i, W_1)$ denote the multiplicity of a pole of a function ϕ_i at w_1, and let ϕ_{ij} denote the jth coefficient in the Laurent expansion of ϕ_i at w_1 $(i = 1, 2, \dots, \eta)$. Let

$$C_{\pi_1} = \begin{bmatrix} \phi_{11} & \phi_{12} & \cdots & \phi_{1\delta(\phi_1, w_1)} & \cdots & \phi_{\eta 1} & \phi_{\eta 2} & \cdots & \phi_{\eta\delta(\phi_\eta, w_1)} \end{bmatrix},$$

and let

$$A_{\pi_1} = \begin{bmatrix} J_{\delta(\phi_1, w_1)}(w_1) & & & \\ & J_{\delta(\phi_2, w_1)}(w_1) & & \\ & & \ddots & \\ & & & J_{\delta(\phi_\eta, w_1)}(w_1) \end{bmatrix},$$

where $J_j(w_1)$ denotes the Jordan block matrix of the size $j \times j$ with w_1 on the diagonal.

Similarly, construct matrices C_{π_i}, A_{π_i} for each pole w_i of S ($i = 2, 3, \dots, \nu$). Put $C_\pi = [C_{\pi_1} \ C_{\pi_2} \ \cdots \ C_{\pi_\nu}]$, and $A_\pi = \operatorname{diag}(A_{\pi_1}, A_{\pi_2}, \dots, A_{\pi_\nu})$. Let n_π be an integer such that the size of the matrix A_π is $n_\pi \times n_\pi$. It follows from construction that $\sigma(A_\pi) \subset \sigma$. Since the set $\{[\phi_i(z)]_{z=w_1} : i = 1, 2, \dots, \eta\}$ is linearly independent, the pair (C_{π_1}, A_{π_1}) is observable (cf. Theorem 3.1 in [3]). Similarly, the pairs (C_{π_i}, A_{π_i}) ($i = 2, 3, \dots, \nu$) are observable. Hence the pair (C_π, A_π) is observable. It remains to show that (4.1) holds.

It follows from the construction of (C_π, A_π) that for every $x_0 \in \mathbf{C}^{n_\pi \times 1}$ there exists $\phi \in S$ such that $\phi - C_\pi(z - A_\pi)^{-1}x_0 \in \mathscr{R}^{m \times 1}(\sigma)$. We need to show that for every $\phi \in S$ there exists $x_0 \in \mathbf{C}^{n_\pi \times 1}$ such that $\phi - C_\pi(z - A_\pi)^{-1}x_0 \in \mathscr{R}^{m \times 1}(\sigma)$. We may assume without loss of generality $\nu = 1$. Let $\phi \in S$ be a function which has a pole at w_1 of order γ, and let $\phi_1, \phi_2, \dots, \phi_\eta$ be as in the construction of (C_{π_1}, A_{π_1}). Then $[\phi(z)]_{z=w_1} \in \operatorname{span}\{[\phi_1(z)]_{z=w_1}, [\phi_2(z)]_{z=w_1}, \dots, [\phi_i(z)]_{z=w_1}\}$, where i is such that $\delta(\phi_i, w_l) \geqslant \gamma$. Since the columns of $C_{\pi_1}(z - A_{\pi_1})^{-1}$ contain the singular part at w_1 of functions $(z - w_1)^j \phi_k(z)$ $[0 \leqslant j \leqslant \delta(\phi_k, w_1) - 1]$, we can choose a vector $x_1 \in \mathbf{C}^{n_\pi \times 1}$ such that $\delta(\phi(z) - C_\pi(z - A_\pi)^{-1}x_1, w_1) < \gamma$. So the assertion follows by induction on $\delta(\phi)$. ■

The pair (C_π, A_π) reflects the pole structure of the module S. We describe now the annihilator of S, which we denote by S°. Thus,

$$S^\circ = \{ f \in \mathscr{R}^{1 \times m} : fg = 0 \ \forall g \in S \}.$$

Proposition 4.2. *Let σ be a subset of* **C**, *and let S be an $\mathscr{R}(\sigma)$-submodule of $\mathscr{R}^{m \times 1}$. Then there exists a matrix polynomial P_κ, without zeros in* **C**, *whose rows span S° (over $\mathscr{R}$) and are orthogonal at infinity. Moreover, if S has a finite number of poles and no essential singularities, and (C_π, A_π) is such that* (4.1) *holds, then the function $P_\kappa(z)C_\pi(z - A_\pi)^{-1}$ is analytic on* **C**.

Proof. Since S° is a subspace of $\mathscr{R}^{1 \times m}$, we can find a minimal polynomial basis $\{v_1, v_2, \dots, v_k\}$ for S°. Let P_κ be the matrix polynomial with rows $v_1, v_2, \dots, v_k$. The orthogonality of the columns of P_κ at infinity, and the fact that P_κ has no zeros in **C**, follow from the properties of minimal polynomial bases proved in [18]. To see that the function $P_\kappa(z)C_\pi(z - A_\pi)^{-1}$ is analytic on **C**, note that each column of $C_\pi(z - A_\pi)^{-1}$ contains the singular part at a pole of S of some function $\phi \in S$, and, by the definition of P_κ, $P_\kappa \phi = 0$. ■

Let σ and S be as above, and let $S(\lambda)$ denote the complex vector space formed by the values at a point λ of functions in S which are analytic at λ. We shall say that S has a *zero* at λ if $\lambda \in \sigma$ and

$$\dim S(\lambda) < \max_{z \in \mathbf{C}_\infty} \dim S(z).$$

PROPOSITION 4.3. *Let σ be a subset of $\mathbf{C}$, and let S be an $\mathscr{R}(\sigma)$-submodule of $\mathscr{R}^{m \times 1}$ with a finite number of zeros. Then there exists a controllable pair of matrices (A_ζ, B_ζ) such that $\sigma(A_\zeta) \subset \sigma$ and, if $P_\kappa \in \mathscr{R}^{k \times m}$ is a matrix polynomial as in Proposition 4.3 and $\sigma(A_\zeta) = \{\lambda_1, \lambda_2, \dots, \lambda_r\}$, the pair*

$$\left(\begin{bmatrix} A_\zeta & & & & \\ & \lambda_1 I_k & & & \\ & & \lambda_2 I_k & & \\ & & & \ddots & \\ & & & & \lambda_r I_k \end{bmatrix}, \begin{bmatrix} B_\zeta \\ P_\kappa(\lambda_1) \\ P_\kappa(\lambda_2) \\ \vdots \\ P_\kappa(\lambda_r) \end{bmatrix} \right) \tag{4.2}$$

is controllable and

$$S \cap \mathscr{R}^{m \times 1}(\sigma) = \Big\{ h \in \mathscr{R}^{m \times 1}(\sigma) :$$

$$P_\kappa h = 0 \text{ and } \sum_{z_0 \in \sigma} \operatorname{Res}_{z = z_0} (z - A_\zeta)^{-1} B_\zeta h(z) = 0 \Big\}. \tag{4.3}$$

Proof. In order to construct the pair (A_ζ, B_ζ), we adapt to the module setting the construction of a canonical set of null functions for a given regular rational matrix functions in [3]. By Corollary 2.6 in [12], there exists Ξ such that $\mathscr{R}^{1 \times m} = S^\circ \oplus_\sigma \Xi$. Let $z_1, z_2, \dots, z_\mu$ be the zeros of S. Using the same notation as in [12], we will denote by $\|\phi\|_{z=\lambda}$ the number 0 if $\phi = 0$, or, if $\phi \neq 0$, the number e^γ, where γ is an integer such that $(z - \lambda)^\gamma \phi(z)$ is analytic and nonzero at λ. Proceeding as in the construction of (C_π, A_π) above, choose functions $\phi_1, \phi_2, \dots, \phi_n \in S$ such that

(a) $[\phi_1(z)]_{z=z_1}, [\phi_2(z)]_{z=z_1}, \dots, [\phi_n(z)]_{z=z_1}$ are linearly independent,
(b) $\sum_{i=1}^n \|\phi_i\|_{z=z_1}$ is maximal subject to condition (a),
(c) $\|\phi_1\|_{z=z_1} \leqslant \|\phi_2\|_{z=z_1} \leqslant \cdots \leqslant \|\phi_n\|_{z=z_1}$.

Let η be such that ϕ_i has a zero at z_1 if and only if $1 \leqslant i \leqslant \eta$. Choose a Smith-McMillan factorization EDF of the rational matrix function $[\phi_1 \ \phi_2 \ \cdots \ \phi_n]$, and let ψ_i be the projection of the ith row of E^{-1} onto Ξ along S° $(i = 1,2,\dots,\eta)$. By Proposition 4.6 in [12], $\psi_1, \psi_2, \dots, \psi_\eta$ are analytic at z_1. Let l_i be the multiplicity of the zero of ϕ_i at z_1 $(i = 1,2,\dots,\eta)$. For $i = 1,2,\dots,\eta$ let ψ_{ij} be the jth coefficient in the Taylor expansion of ψ_i at z_1 (that is, $\psi_i(z) = \sum_{j=1}^{\infty} z^{j-1}\psi_{ij}$). We put

$$A_{\zeta_1} = \begin{bmatrix} J_{l_1}(z_1) & & & \\ & J_{l_2}(z_1) & & \\ & & \ddots & \\ & & & J_{l_\eta}(z_1) \end{bmatrix} \quad \text{and} \quad B_{\zeta_1} = \begin{bmatrix} \psi_{1l_1} \\ \vdots \\ \psi_{12} \\ \psi_{11} \\ \vdots \\ \psi_{\eta l_\eta} \\ \vdots \\ \psi_{\eta 2} \\ \psi_{\eta 1} \end{bmatrix}.$$

Similarly, construct a pair of matrices $(A_{\zeta_i}, B_{\zeta_i})$ for the zero z_i of S $(i = 2,3,\dots,\mu)$. We put

$$A_\zeta = \begin{bmatrix} A_{\zeta_1} & & & \\ & A_{\zeta_2} & & \\ & & \ddots & \\ & & & A_{\zeta_\mu} \end{bmatrix}, \qquad B_\zeta = \begin{bmatrix} B_{\zeta_1} \\ B_{\zeta_2} \\ \vdots \\ B_{\zeta_\mu} \end{bmatrix}.$$

Since $\sigma(A_\zeta) = \{z_1, z_2, \dots, z_\mu\}$ is the set of zeros of S, $\sigma(A_\zeta) \subset \sigma$. To show that the pair (A_ζ, B_ζ) is controllable, we assume without loss of generality that $\mu = 1$. Now the pair $(A_{\zeta_1}, B_{\zeta_1})$ is controllable if and only if the vectors $\psi_1(z_1), \psi_2(z_1), \dots, \psi_\eta(z_1)$ are linearly independent. But, in view of Proposition 4.4 in [12], $\psi_1(z_1), \psi_2(z_1), \dots, \psi_\eta(z_1)$ are projections of the first η rows of $E^{-1}(z_1)$ along the subspace of $\mathbf{C}^{1\times m}$ equal to the span of the last $m-n$ rows of $E^{-1}(z_1)$. So they must be linearly independent. Thus, the pair (A_ζ, B_ζ) is controllable. Also, by the choice of Ξ, the pair (4.2) is controllable. It remains to show that (4.3) holds.

Let $h \in S \cap \mathscr{R}^{m\times 1}(\sigma)$. Then, plainly, $P_\kappa h = 0$. We show that $(z - A_\zeta)^{-1}B_\zeta h(z)$ is analytic on σ. For simplicity, we will show that $(z -$

$A_{\zeta_1})^{-1}B_{\zeta_1}h(z)$ is analytic at z_1. Since the rows of $(z-A_{\zeta_1})^{-1}B_{\zeta_1}$ contain the singular parts at z_1 of functions $(z-z_1)^{-j}\psi_i(z)$ $(1\leqslant i\leqslant\eta,\ 1\leqslant j\leqslant l_i)$, it suffices to show that $\psi_i(z)h(z)$ vanishes at z_1 to order at least l_i $(i=1,2,\dots,\eta)$. If h vanishes at z_1 to order at least l_1, there is nothing to prove. Let $1\leqslant\gamma\leqslant\eta$ be the largest integer such that h vanishes at z_1 to order less than l_γ. By the construction, $[h(z)]_{z=z_1}$ is contained in span$\{[\phi_j(z)]_{z=z_1}, [\phi_{j+1}(z)]_{z=z_1},\dots,[\phi_n(z)]_{z=z_1}\}$, where $j>\gamma$ is such that h vanishes at z_1 to order at least l_j. Choose numbers $\alpha_1,\alpha_2,\dots,\alpha_\eta$ and integers $\beta_1,\beta_2,\dots,\beta_\eta$ such that

$$h_1(z)=h(z)-\sum_{i=j}^{n}\alpha_i(z-z_1)^{\beta_i}\phi_i$$

vanishes at z_1 to a greater order than $h(z)$ does. Then ψ_ih vanishes at z_1 to the same order as ψ_ih_1 $(i=1,2,\dots,\gamma)$. It follows by induction that ψ_ih vanishes at z_1 to order l_i $(i=1,2,\dots,\eta)$. Thus, $\Sigma_{z_0\in\sigma}\operatorname{Res}_{z=z_0}(z-A_\zeta)^{-1}B_\zeta h(z)=0$, and one of the two inclusions claimed in (4.3) is established.

Suppose now $h\in\mathscr{R}^{m\times 1}(\sigma)$ is such that $Ph=0$ and

$$\sum_{z_0\in\sigma}\operatorname{Res}_{z=z_0}(z-A_\zeta)^{-1}B_\zeta h(z)=0.$$

We need to show $h\in S$. Let EDF be the Smith-McMillan factorization of the function $[\phi_1\ \phi_2\ \cdots\ \phi_n]$ introduced above. Then $k+n=m$, and the bottom k rows of E^{-1} span S°. Indeed, each of the first n columns of E is contained in S, so S° is a subset of the span of the bottom $m-n$ rows of E^{-1}. Since the last $m-n$ columns of E do not contain functions from S, each of the bottom $m-n$ rows of E^{-1} is contained in S°. Thus, S° is equal to the span of the bottom $m-n$ rows of E^{-1}, as asserted. Consequently, h is a linear combination over $\mathscr{R}$ of the first n columns of E. Call the columns of the function ED $W_1,W_2,\dots,W_n$. Then $W_1,W_2,\dots,W_n\in S$, and, after dividing each W_i by a scalar polynomial, we may assume $(z-\lambda)^{-1}W_i(z)\notin S$ for all $i=1,2,\dots,n$ and all $\lambda\in\sigma$. Let $r_1,r_2,\dots,r_n\in\mathscr{R}$ be such that $h=\Sigma_{i=1}^{N}r_iW_i$. We assert that $r_1,r_2,\dots,r_n\in\mathscr{R}(\sigma)$. Indeed, since $h\in\mathscr{R}^{m\times 1}(\sigma)$, r_i is analytic at $\lambda\in\sigma$ whenever W_i does not have a zero at λ $(i=1,2,\dots,n)$. Suppose W_i vanishes at λ for some $i\in\{1,2,\dots,n\}$ and some $\lambda\in\sigma$. Since $[W_1]_{z=\lambda}$, $[W_2]_{z=\lambda},\dots,[W_n]_{z=\lambda}$ are linearly independent and, by dimension count, span $S(\lambda)$, s has a zero at λ. For the sake of definiteness, assume $\lambda=z_2$ and let $\tilde\phi_1,\tilde\phi_2,\dots,\tilde\phi_n$ be the functions used to construct $(A_{\zeta_2},B_{\zeta_2})$. Let $\tilde E\tilde D\tilde F$ be a Smith-McMillan factorization of the function

$[\tilde{\phi}_1 \ \tilde{\phi}_2 \ \cdots \ \tilde{\phi}_n]$, and let $V_1, V_2, \ldots, V_n$ be the columns of $\tilde{E}\tilde{D}$. Then

$$\begin{bmatrix} W_1 & W_2 & \cdots & W_n \end{bmatrix} = \begin{bmatrix} V_1 & V_2 & \cdots & V_n \end{bmatrix} Q$$

for some regular rational matrix function Q such that $Q, Q^{-1} \in \mathscr{R}^{n\times n}(\{z_2\})$. Then, if r_i has a pole at z_2 for some $i \in \{1,2,\ldots,n\}$,

$$\operatorname{Res}_{z=z_2}(z-A_{\zeta_2})^{-1}B_{\zeta_2}\left(\begin{bmatrix} W_1(z) & W_2(z) & \cdots & W_n(z) \end{bmatrix}\begin{bmatrix} r_1(z) \\ r_2(z) \\ \vdots \\ r_n(z) \end{bmatrix}\right)$$

$$= \operatorname{Res}_{z=z_2}(z-A_{\zeta_2})^{-1}B_{\zeta_2}\left(\begin{bmatrix} V_1(z) & V_2(z) & \cdots & V_n(z) \end{bmatrix}Q(z)\begin{bmatrix} r_1(z) \\ r_2(z) \\ \vdots \\ r_n(z) \end{bmatrix}\right)$$

$$\neq 0.$$

Consequently,

$$\sum_{z_0 \in \sigma} \operatorname{Res}_{z=z_0}(z-A_\zeta)^{-1}B_\zeta h(z) \neq 0,$$

a contradiction. Thus $r_i \in \mathscr{R}(\sigma)$ for $i = 1,2,\ldots,n$, and, since h is a linear combination of functions in S with coefficients equal to $r_1, r_2, \ldots, r_n$, it follows that $h \in S$. So the equality (4.3) holds. ■

We show now that any $\mathscr{R}(\sigma)$-submodule of $\mathscr{R}^{m\times 1}$ which has a finite number of zeros and poles, and does not have essential singularities, can be described as in (2.11).

THEOREM 4.4. *Let σ be a subset of* **C**, *and let S be an $\mathscr{R}(\sigma)$-submodule of $\mathscr{R}^{m\times 1}$ with a finite number of poles and zeros and without essential singularities. Then there exist nonnegative integers n_π, n_ζ, k; matrices $C_\pi \in$ $\mathbf{C}^{m\times n_\pi}$, $A_\pi \in \mathbf{C}^{n_\pi \times n_\pi}$, $A_\zeta \in \mathbf{C}^{n_\zeta\times n_\zeta}$, $B_\zeta \in C^{n_\zeta \times m}$, $\Gamma \in \mathbf{C}^{n_\zeta\times n_\pi}$; and a matrix* *polynomial* $P_\kappa \in \mathscr{R}^{k\times m}$ *such that*

$$S = \left\{ C_\pi(z-A_\pi)^{-1}x + h(z) : x \in \mathbf{C}^{n_\pi \times 1},\ h \in \mathscr{R}^{m\times 1}(\sigma),\ \text{and} \right.$$

$$\left. \sum_{z_0\in\sigma} \operatorname{Res}_{z=z_0}(z-A_\zeta)^{-1}B_\zeta h(z) = \Gamma x \right\}$$

$$\cap \{ f \in \mathscr{R}^{m\times 1} : P_n f = 0 \}, \tag{4.4}$$

and

(i) *the pair* (C_π, A_π) *is observable and* $\sigma(A_\pi) \subset \sigma$;

(ii) *the pair* (A_ζ, B_ζ) *is controllable and* $\sigma(A_\zeta) \subset \sigma$;

(iii) *the matrix polynomial* P_κ *has no zeros in* $\mathbf{C}$, *and the rows of* P_κ *are orthogonal at infinity*;

(iv) *the rational matrix function* $P_\kappa(z)C_\pi(z-A_\pi)^{-1}$ *is analytic on* $\mathbf{C}$;

(v) *if* $\sigma(A_\zeta) = \{\lambda_1, \lambda_2, \dots, \lambda_r\}$ *and* I_k *is the* $k\times k$ *identity matrix, the pair*

$$\left(\begin{bmatrix} A_\zeta & & & & \\ & \lambda_1 I_k & & & \\ & & \lambda_2 I_k & & \\ & & & \ddots & \\ & & & & \lambda_r I_k \end{bmatrix}, \begin{bmatrix} B_\zeta \\ P_\kappa(\lambda_1) \\ P_\kappa(\lambda_2) \\ \vdots \\ P_\kappa(\lambda_r) \end{bmatrix} \right)$$

is controllable;

(vi) $\Gamma A_\pi - A_\zeta \Gamma = B_\zeta C_\pi$.

Proof. By Propositions 4.1–4.3, there exist pairs of matrices (C_π, A_π) and a matrix polynomial P_κ which satisfy conditions (i)–(v). We define now the matrix Γ. Let $e_i \in \mathbf{C}^{n_\pi\times 1}$ be the vector with 1 in the ith position and 0's elsewhere. It follows from the construction of (C_π, A_π) that for every $i = 1, 2, \dots, n_\pi$ there exists a function $\phi_i \in \mathscr{R}^{m\times 1}(\sigma)$ such that $C_\pi(z - A_\pi)^{-1}e_i + \phi_i(z) \in S$. Let

$$\Gamma = \sum_{z_0\in\sigma} \operatorname{Res}_{z=z_0}(z-A_\zeta)^{-1}B_\zeta\begin{bmatrix}\phi_1(z) & \phi_2(z) & \cdots & \phi_{n_\pi}(z)\end{bmatrix}.$$

If we choose functions $\tilde{\phi}_i \in \mathscr{R}^{m\times 1}(\sigma)$ $(i = 1, 2, \dots, n_\pi)$ such that $C_\pi(z -$

$A_\pi)^{-1}e_i + \tilde{\phi}_i(z) \in S$, then, for $i = 1, 2, \ldots, n_\pi$, $\phi_i - \tilde{\phi}_i \in S \cap \mathscr{R}^{m \times 1}(\sigma)$. So, by Proposition 4.3,

$$\sum_{z_0 \in \sigma} \operatorname{Res}_{z=z_0} (z - A_\zeta)^{-1} B_\zeta \begin{bmatrix} \phi_1(z) - \tilde{\phi}_1(z) & \cdots & \phi_{n_\pi}(z) - \tilde{\phi}_{n_\pi}(z) \end{bmatrix} = 0,$$

and Γ is well defined.

We show that Γ satisfies the equality (4.4). Let $f \in S$. Then, by Proposition 4.1, there exists a vector $x_f \in \mathbf{C}^{n_\pi \times 1}$ and a function $h_f \in \mathscr{R}^{m \times 1}(\sigma)$ such that

$$f(z) = C_\pi (z - A_\pi)^{-1} x_f + h_f(z).$$

By the definition of Γ,

$$\sum_{z_0 \in \sigma} \operatorname{Res}_{z=z_0} (z - A_\zeta)^{-1} B_\zeta h_f(z) = \Gamma x_f.$$

Plainly, $P_\kappa f = 0$. Thus, S is a subset of the right-hand side of (4.4).

Suppose now a function $f(z) = C_\pi(z - A_\pi)^{-1}x_f + h_f(z)$ is a member of the right-hand side of (4.4). Then

$$\sum_{z_0 \in \sigma} \operatorname{Res}_{z=z_0} (z - A_\zeta)^{-1} B_\zeta h_f(z) = \Gamma x_f.$$

By the construction of (C_π, A_π), there exists a function $\phi(z) = C_\pi(z - A_\pi)^{-1}x_f + h_\phi(z) \in S$. Since S is contained in the right-hand side of (4.4),

$$\sum_{z_0 \in \sigma} \operatorname{Res}_{z=z_0} (z - A_\zeta)^{-1} B_\zeta h_\phi(z) = \Gamma x_f.$$

Therefore

$$\sum_{z_0 \in \sigma} \operatorname{Res}_{z=z_0} (z - A_\zeta)^{-1} B_\zeta \left[h_f(z) - h_\phi(z) \right] = 0.$$

Since $P_\kappa f = 0$ and $P_\kappa \phi = 0$, $P_\kappa(h_f - h_\phi) = 0$, and it follows by Proposition 4.3 that $f - \phi \in S$. So $f \in S$ and (4.4) is established.

We show now that Γ satisfies condition (vi). Let $x \in \mathbf{C}^{n_\pi \times 1}$ be arbitrary. By the construction of (C_π, A_π), there exists a function $h \in \mathscr{R}^{m \times 1}(\sigma)$ such that $k(z) = C_\pi(z - A_\pi)^{-1}x + h(z) \in S$. Since S is an $\mathscr{R}(\sigma)$-module, $zk(z) = C_\pi(z - A_\pi)^{-1}A_\pi x + C_\pi x + zh(z)$ is a member of S. Hence, by (4.4),

$$\sum_{z_0 \in \sigma} \operatorname{Res}_{z=z_0}(z - A_\zeta)^{-1}B_\zeta h(z) = \Gamma x \tag{4.5}$$

and

$$\sum_{z_0 \in \sigma} \operatorname{Res}_{z=z_0}(z - A_\zeta)^{-1}B_\zeta[C_\pi x + zh(z)] = \Gamma A_\pi x. \tag{4.6}$$

Since $\sigma(A_\zeta) \in \sigma$, the equality (4.6) implies

$$\begin{aligned} B_\zeta C_\pi &= \Gamma A_\pi x - \sum_{z_0 \in \sigma} \operatorname{Res}_{z=z_0}(z - A_\zeta)^{-1}B_\zeta z h(z) \\ &= \Gamma A_\pi x - \left(\sum_{z_0 \in \sigma} \operatorname{Res}_{z=z_0}(z - A_\zeta)^{-1}A_\zeta B_\zeta h(z) - \sum_{z_0 \in \sigma} \operatorname{Res}_{z=z_0} B_\zeta h(z) \right) \\ &= \Gamma A_\pi x - A_\zeta \left(\sum_{z_0 \in \sigma} \operatorname{Res}_{z=z_0}(z - A_\zeta)^{-1}B_\zeta h(z) \right). \end{aligned}$$

Substituting (4.5) into the preceding equality, we get

$$B_\zeta C_\pi x = \Gamma A_\pi x - A_\zeta \Gamma x.$$

Since $x \in \mathbf{C}^{n_\pi \times 1}$ was arbitrary, condition (vi) follows. ■

Theorem 4.4 shows that every $\mathscr{R}(\sigma)$-submodule of $\mathscr{R}^{m \times 1}$ with a finite number of poles and zeros, and without essential singularities, can be described as in (4.4). The converse statement is also true.

Theorem 4.5. *Let σ be a subset of* $\mathbf{C}$, *let n_π, n_ζ, k, m be integers, and suppose matrices $C_\pi \in \mathbf{C}^{m \times n_\pi}$, $A_\pi \in \mathbf{C}^{n_\pi \times n_\pi}$, $A_\zeta \in \mathbf{C}^{n_\zeta \times n_\zeta}$, $B_\zeta \in \mathbf{C}^{n_\zeta \times m}$, $\Gamma \in \mathbf{C}^{n_\zeta \times n_\pi}$, and a matrix polynomial $P_\kappa \in \mathscr{R}^{k \times m}$ satisfy conditions* (i)–(vi) *in*

Theorem 4.4. *Then*

$$S=\left\{C_\pi(z-A_\pi)^{-1}x+h(z):x\in \mathbf{C}^{n_\pi\times 1},\ h\in\mathscr{R}^{m\times 1}(\sigma),\ \textit{and}\right.$$

$$\left.\sum_{z_0\in\xi}\operatorname{Res}_{z=z_0}(z-A_\zeta)^{-1}B_\zeta h(z)=\Gamma x\right\}$$

$$\cap\{f\in\mathscr{R}^{m\times 1}:P_k f=0\}$$

is an $\mathscr{R}(\sigma)$*-submodule of* $\mathscr{R}^{m\times 1}$ *with a finite number of poles and zeros and without essential singularities. Moreover,*

(i) $P_{\sigma^c}(S)=\{C_\pi(z-A_\pi)^{-1}x:x\in\mathbf{C}^{n_\pi\times 1}\}$,
(ii) $\operatorname{Ker}P_\kappa$ *is the* $\mathscr{R}$*-linear subspace of* $\mathscr{R}^{m\times 1}$ *generated by* S,
(iii) $S\cap\mathscr{R}^{m\times 1}(\sigma)=\{h\in\mathscr{R}^{m\times 1}(\sigma):P_\kappa h=0$ *and* $\sum_{z_0\in\sigma}\operatorname{Res}_{z=z_0}(z-A_\zeta)^{-1}B_\zeta h(z)=0\}$,
(iv) *there exists* Λ *such that* $\mathscr{R}^{1\times m}=\Lambda\oplus_\sigma(\mathscr{R}^{1\times k}P_\kappa)$ *and*

$$\left\{C_\pi(z-A_\pi)^{-1}x+h(z):x\in C^{n_\pi\times 1},\ h\in\mathscr{R}^{m\times 1}(\sigma),\right.$$

$$\left.\textit{and}\ \sum_{z_0\in\sigma}\operatorname{Res}_{z=z_0}(z-A_\zeta)^{-1}B_\zeta h(z)=\Gamma x\right\}=S\dotplus\left[\Lambda^\circ\cap\mathscr{R}^{m\times 1}(\sigma)\right].$$

Proof. By Theorem 3.2.1 in [4], the set

$$\left\{C_\pi(z-A_\pi)^{-1}x+h(z):x\in\mathbf{C}^{n_\pi\times 1},\ h\in\mathscr{R}^{m\times 1}(\sigma),\ \text{and}\right.$$

$$\left.\sum_{z_0\in\sigma}\operatorname{Res}_{z=z_0}(z-A_\zeta)^{-1}B_\zeta h(z)=\Gamma x\right\}$$

is an $\mathscr{R}(\sigma)$-submodule of $\mathscr{R}^{m\times 1}$. Since each $\mathscr{R}$-linear subspace of $\mathscr{R}^{m\times 1}$ is, in particular, an $\mathscr{R}(\sigma)$-submodule of $\mathscr{R}^{m\times 1}$, S is an intersection of two $\mathscr{R}(\sigma)$-submodules of $\mathscr{R}^{m\times 1}$. So S is an $\mathscr{R}(\sigma)$-submodule of $\mathscr{R}^{m\times 1}$. Since poles of S are eigenvalues of A_π, and zeros of S are eigenvalues of A_ζ, S has a finite number of poles and zeros. Plainly, S has no essential singularities.

To see that (ii) holds, let $\phi_1, \phi_2, \dots, \phi_n$ be the functions which span $\operatorname{Ker} P_\kappa$. Then there exist scalar polynomials $q_1, q_2, \dots, q_n$ such that $q_i\phi_i \in \mathscr{R}^{m\times 1}(\sigma)$ and

$$(z - A_\zeta)^{-1} B_\zeta q_i \phi_i(z) \in \mathscr{R}^{n_\zeta \times 1}(\sigma)$$

for all $i = 1, 2, \dots, n$. So $\{q_i\phi_i : 1 \leqslant n\} \subset S$.

We now show (iv). By Theorem 4.2 and Lemma 3.16 in [12], there exists Λ such that $\mathscr{R}^{1\times m} = \Lambda \oplus_\sigma (\mathscr{R}^{1\times k} P_\kappa)$ and the pair (A_ζ, B_ζ) is left similar to a pair constructed from Taylor coefficients of functions in Λ. Hence, if $x \in \mathbf{C}^{1\times n_\zeta}$, there exists $h \in \Lambda$ such that $h - x(z - A_\zeta)^{-1} B_\zeta \in \mathscr{R}^{1\times m}(\sigma)$. If $f \in \Lambda^\circ \cap \mathscr{R}^{m\times 1}(\sigma)$, then $hf = 0$ and so $x(z - A_\zeta)^{-1} B_\zeta f(z) \in \mathscr{R}(\sigma)$. Consequently,

$$\begin{aligned} &S \dot{+} \left[\Lambda^\circ \cap \mathscr{R}^{m\times 1}(\sigma)\right] \\ &\quad \subset \Bigg\{ C_\pi (z - A_\pi)^{-1} x + h(z) : x \in \mathbf{C}^{n_\pi \times 1},\ h \in \mathscr{R}^{m\times 1}(\sigma), \\ &\qquad\qquad \text{and } \sum_{z_0 \in \sigma} \operatorname{Res}_{z = z_0} (z - A_\zeta)^{-1} B_\zeta h(z) = \Gamma x \Bigg\}. \end{aligned} \tag{4.7}$$

Let f be a member of the set on the right-hand side of (4.7). By Proposition 4.6 in [12], $f = f_{\mathrm{Ker}} + f_{\Lambda^\circ}$, where $f_{\mathrm{Ker}} \in \operatorname{Ker} P_\kappa$ and $f_{\Lambda^\circ} \in \Lambda^\circ \cap \mathscr{R}^{m\times 1}(\sigma)$. Then

$$\sum_{z_0 \in \sigma} \operatorname{Res}_{z = z_0} (z - A_\zeta)^{-1} B_\zeta f_{\Lambda^\circ}(z) = 0$$

and hence $f_{\mathrm{Ker}} \in S$. It follows that the inclusion (4.7) is an equality.

We now show (ii). It follows immediately from the definition of S that

$$P_{\sigma^c}(S) \subset \left\{ C_\pi (z - A_\pi)^{-1} x : x \in \mathbf{C}^{n_\pi \times 1} \right\}. \tag{4.8}$$

Let $x \in \mathbf{C}^{n_\pi \times 1}$ be arbitrary. By Lemma 3.2.2 in [4], there exists $h_x \in \mathscr{R}^{m\times 1}(\sigma)$ such that

$$\sum_{z_0 \in \sigma} \operatorname{Res}_{z = z_0} (z - A_\zeta)^{-1} B_\zeta h_x(z) = \Gamma x.$$

By (iv),

$$C_\pi(z - A_\pi)^{-1}x + h_x(z) = f_{\mathrm{Ker}} + f_{\Lambda^\circ},$$

where $f_{\Lambda^\circ} \in \Lambda^\circ \cap \mathscr{R}^{m\times 1}(\sigma)$ and $f_{\mathrm{Ker}} \in S$. So

$$P_{\sigma^c}(f_{\mathrm{Ker}}) = C_\pi(z - A_\pi)^{-1}x,$$

and (ii) follows.

It remains to show (iii). By Theorem 3.4.1 and Lemma 3.3.7 in [4],

$$\begin{aligned}\mathscr{R}^{m\times 1}(\sigma) \cap \Big\{ C_\pi(z - A_\pi)^{-1}x + h(z) : x \in \mathbf{C}^{n_\pi\times 1},\ h \in \mathscr{R}^{m\times 1}(\sigma), \text{ and} \\ \sum_{z_0\in\sigma} \mathrm{Res}_{z=z_0}(z - A_\zeta)^{-1}B_\zeta h(z) = \Gamma x \Big\} \\ = \Big\{ h \in \mathscr{R}^{m\times 1}(\sigma) : \sum_{z_0\in\sigma} \mathrm{Res}_{z=z_0}(z - A_\zeta)^{-1}B_\zeta h(z) = 0 \Big\}.\end{aligned}$$

So (iii) follows from (iv). ■

The results of this section for the regular case are presented in Chapter 14 of [6]. There the hypothesis that the module has a finite number of poles and zeros, and no essential singularities, is expressed in terms of dimensions of vector spaces over **C**. Indeed, S has a finite number of poles and no essential singularities if and only if

$$\dim_{\mathbf{C}}(P_{\sigma^c}S) < \infty,$$

where P_{σ^c} denotes the projection of $\mathscr{R}^{m\times 1}$ onto $\mathscr{R}_0^{m\times 1}(\sigma^c)$ along $\mathscr{R}^{m\times 1}(\sigma)$. The module S has a finite number of zeros if and only if

$$\dim_{\mathbf{C}} \frac{\Omega \cap \mathscr{R}^{m\times 1}(\sigma)}{S \cap \mathscr{R}^{m\times 1}(\sigma)} < \infty,$$

where Ω is the subspace of $\mathscr{R}^{m\times 1}$ generated by S. Also, the pole pair (C_π, A_π) and the null pair (A_ζ, B_ζ) are defined in a more coordinate-free form as linear transformations with A_π acting on the pole module and A_ζ acting on the zero module.

5. EXISTENCE OF FUNCTIONS WITH A GIVEN NULL-POLE SUBSPACE

Theorem 4.4 and Theorem 4.5 imply that $\mathscr{R}(\sigma)$-submodules of $\mathscr{R}^{m\times 1}$ with a finite number of poles and zeros, and without essential singularities, are precisely the sets described as in the formula (4.4) by matrices and a matrix polynomial which satisfy conditions (i)–(vi). By Proposition 4.1 in [12] and by Theorem 3.1 above, the left spectral triple $\tau_s = ((C_\pi, A_\pi), (A_\zeta, B_\zeta), \Gamma)$ of a rational matrix function W over σ, and a matrix polynomial P_κ associated with a left kernel triple $\tau_\kappa = (A_\kappa, B_\kappa, D_\kappa)$, necessarily satisfy conditions (i)–(vi) in Theorem 4.4. Therefore the question of the existence of a rational matrix function with a given $\mathscr{R}(\sigma)$-submodule of $\mathscr{R}^{m\times 1}$ as a left null-pole subspace over σ can be reformulated as follows: when does there exist a rational matrix function W with τ_s as a left spectral triple over σ and τ_κ as a left kernel triple? Before answering this question, we need to prove the following auxiliary result.

LEMMA 5.1. *Let* $\sigma \subset \mathbf{C}$, *and suppose* $\tau_s = ((C_\pi, A_\pi), (A_\zeta, B_\zeta), \Gamma)$ *and a matrix polynomial* P_κ *satisfy conditions* (i)–(vi) *in Theorem* 4.4. *Let*

$$S = \Big\{ C_\pi(z - A_\pi)^{-1}x + h(z) : x \in \mathbf{C}^{n_\pi \times 1},\ h \in \mathscr{R}^{m\times 1}(\sigma),\ \text{and} \\ \sum_{z_0 \in \sigma} \operatorname{Res}_{z=z_0}(z - A_\zeta)^{-1} B_\zeta h(z) = \Gamma x \Big\} \\ \cap \{ f \in \mathscr{R}^{m\times 1} : P_\kappa f = 0 \},$$

and let W be a rational matrix function with S as a left null-pole subspace over σ. Then τ_s is a left spectral triple for W over σ, and P_κ is a left kernel polynomial for W.

Proof. We assume without loss of generality that W has full column rank. The fact that P_κ is a left kernel polynomial for W follows from Theorem

4.5. To show the other assertion, let Λ be as in Theorem 4.5. Choose a minimal polynomial basis $\{v_1, v_2, \ldots, v_k\}$ for Λ°, and let

$$W_E = \begin{bmatrix} W & v_1 & v_2 & \cdots & v_k \end{bmatrix}.$$

By Theorems 3.4.1 and 3.4.2 in [4], τ_s is a left σ-spectral triple for W_E. Hence, plainly, (C_π, A_π) is a right pole pair for W over σ, and (A_ζ, B_ζ) is a left null pair for W over σ. So, by Lemma 3.2 above, τ_s is a left σ-spectral triple for W. ■

We can now prove the following result.

THEOREM 5.2. *Let* $\sigma \subset \mathbf{C}$, *and suppose matrices* $C_\pi \in \mathbf{C}^{m \times n_\pi}$, $A_\pi \in \mathbf{C}^{n_\pi \times n_\pi}$, $A_\zeta \in \mathbf{C}^{n_\zeta \times n_\zeta}$, $B_\zeta \in \mathbf{C}^{n_\zeta \times m}$, $A_\kappa \in \mathbf{C}^{n_\kappa \times n_\kappa}$, $B_\kappa \in \mathbf{C}^{n_\kappa \times m}$, $D_\kappa \in \mathbf{C}^{k \times m}$ *are such that*

(i) *the pair* (C_π, A_π) *is observable and* $\sigma(A_\pi) \subset \sigma$,

(ii) *the pair* (A_ζ, B_ζ) *is controllable and* $\sigma(A_\zeta) \subset \sigma$,

(iii) *the pair* (A_κ, B_κ) *is controllable*, A_κ *is a nilpotent Jordan matrix with at most k blocks, and the matrix polynomial* P_κ *associated with* $(A_\kappa, B_\kappa, D_\kappa)$ *has the Smith-McMillan form* $[I \;\; 0]$,

(iv) *the function* $P_\kappa(z) C_\pi (z - A_\pi)^{-1}$ *is analytic on* $\mathbf{C}$,

(v) *if* $\sigma(A_\zeta) = \{\lambda_1, \lambda_2, \ldots, \lambda_r\}$ *and* I_κ *is the* $k \times k$ *identity matrix, the pair*

$$\left(\begin{bmatrix} A_\zeta & & & & \\ & \lambda_1 I_\kappa & & & \\ & & \lambda_2 I_\kappa & & \\ & & & \ddots & \\ & & & & \lambda_r I_\kappa \end{bmatrix}, \begin{bmatrix} B_\zeta \\ P_\kappa(\lambda_1) \\ P_\kappa(\lambda_2) \\ \vdots \\ P_\lambda(\lambda_r) \end{bmatrix} \right)$$

is controllable.

Let Γ *be a solution of the Sylvester equation*

$$XA_\pi - A_\zeta X = B_\zeta C_\pi.$$

Then there exists a left-invertible function $W \in \mathscr{R}^{m \times (m-\kappa)}$ *with* $(A_\kappa, B_\kappa, D_\kappa)$ *as a left kernel triple and* $\tau_s = ((C_\pi, A_\pi), (A_\zeta, B_\zeta), \Gamma)$ *as a left* σ*-spectral triple.*

Proof. We may assume $\sigma(A_\pi) \cup \sigma(A_\zeta) \neq \emptyset$. Let P be a matrix polynomial associated with the triple $(A_\kappa, B_\kappa, D_\kappa)$. For the sake of definiteness, we assume that the size of P is $k \times m$. Define $S_{W\sigma}$ by

$$S_{W\sigma} = \left\{ C_\pi(z - A_\pi)^{-1}x + h(z) : h \in \mathscr{R}^{m \times 1}(\sigma) \right.$$
$$\left. \text{and } \sum_{z_0 \in \sigma} \operatorname{Res}_{z=z_0}(z - A_\zeta)^{-1} B_\zeta h(z) = \Gamma x \right\}$$
$$\cap \left\{ f \in \mathscr{R}^{m \times 1} : Pf = 0 \right\}. \tag{5.1}$$

By Theorem 4.5, $S_{W\sigma}$ is an $\mathscr{R}(\sigma)$-module. Clearly $S_{W\sigma}$ is a submodule of the $\mathscr{R}(\sigma)$-module

$$S_\sigma = \left\{ C_\pi(z - A_\pi)^{-1}x + h(z) : h \in \mathscr{R}^{m \times 1}(\sigma) \right.$$
$$\left. \text{and } \sum_{z_0 \in \sigma} \operatorname{Res}_{z=z_0}(z - A_\zeta)^{-1} B_\zeta h(z) = \Gamma x \right\}.$$

By Theorem 5.4.1 in [4], the module S_σ is a finitely generated free module. Since, by Proposition 1 in [24], $\mathscr{R}(\sigma)$ is a principal-ideal domain, $S_{W\sigma}$ is a finitely generated free module (cf. [21, Theorem 3.7 in Chapter 3]). Choose a basis $\{w_1, w_2, \dots, w_n\}$ for $S_{W\sigma}$, and define the rational matrix function W by

$$W = [\, w_1 \quad w_2 \quad \cdots \quad w_n \,].$$

Then, by Lemma 5.1, τ_s is a left σ-spectral triple and τ_κ is a left kernel triple for W. Since $\{w_1, w_2, \dots, w_n\}$ is a basis for a free module over $\mathscr{R}(\sigma)$, W has full column rank over $\mathscr{R}$. ■

It is worthwhile to point out that Theorem 5.2 above is stronger than Theorem 4.2 in [12]. In fact, in view of Theorem 3.3, Theorem 5.2 has the following corollary.

COROLLARY 5.3. *Let σ be a subset of* **C**, *let (C_π, A_π), (A_ζ, B_ζ), and $(A_\kappa, B_\kappa, D_\kappa)$ be collections of matrices which satisfy conditions* (i)–(v) *in Proposition* 4.1 *in* [12], *and let $\mathscr{E}$ be the set of rational matrix functions with (C_π, A_π) as a right pole pair over σ, (A_ζ, B_ζ) as a left null pair over σ, and*

$(A_\kappa, B_\kappa, D_\kappa)$ as a left kernel triple. Divide $\mathscr{E}$ into classes of functions with the same left null-pole subspaces over σ. Then the classes in $\mathscr{E}$ are in one-to-one correspondence with the set of solutions of the equation $XA_\pi - A_\zeta X = B_\zeta C_\pi$.

We can now give a criterion for determining which $\mathscr{R}(\sigma)$-submodules of $\mathscr{R}^{m\times 1}$ are left null-pole subspaces of rational matrix functions.

THEOREM 5.4. *Let $\sigma \subset \mathbf{C}$, and let S be an $\mathscr{R}(\sigma)$-submodule of $\mathscr{R}^{m\times 1}$. Then there exists a function $W \in \mathscr{R}^{m\times n}$ such that $S = \mathscr{S}_\sigma(W)$ if and only if S has a finite number of poles and zeros and no essential singularities.*

Proof. Sufficiently follows from Theorems 4.4 and 5.2. Since every rational matrix function has a finite number of poles and zeros in $\mathbf{C}_\infty$, and since every pole of a rational matrix function has a finite order, the condition stated in the theorem is also necessary. ∎

6. CONSTRUCTION OF FUNCTIONS WITH A GIVEN LEFT NULL-POLE SUBSPACE

We show now that the null-pole coupling operator is preserved in the construction of a rational matrix function with a given left zero-pole structure developed in [12]. This construction produces a left-invertible rational matrix function W, with (C_π, A_π) as a right pole pair over a subset σ of $\mathbf{C}$, (A_ζ, B_ζ) as a left null pair over σ, and $(A_\kappa, B_\kappa, D_\kappa)$ as a left kernel triple, from a regular rational matrix function with (C_π, A_π) as a right pole pair over σ, and (A_ζ, B_ζ) as a left null pair over σ. The construction can be summarized as follows.

CONSTRUCTION 6.1. Let $\sigma \subset \mathbf{C}$, and let $H \in \mathscr{R}^{m\times m}$ be a regular function with $\tau_s = ((C_\pi, A_\pi), (A_\zeta, B_\zeta), \Gamma)$ as a left spectral triple over σ. Let $\tau_\kappa = (A_\kappa, B_\kappa, D_\kappa)$ be a triple of matrices whose associated matrix polynomial $P \in \mathscr{R}^{k\times m}$ satisfies conditions (iii)–(v) in Theorem 4.4. Let $n = m - k$. Construct a function $W \in \mathscr{R}^{m\times n}$ as follows.

Step 1. Choose a Smith-McMillan factorization EDF of H, and put $W_1 = ED$.

Step 2. Let ν be the largest geometric multiplicity of a pole of H in σ, let μ be the largest geometric multiplicity of a zero of H in σ, and let η be the largest sum of the geometric multiplicity of a pole and the geometric multiplicity of a zero of H at any single point of σ. Let d_i denote the ith diagonal entry of D. For $i = \eta - \mu + 1$, $\eta - \mu +$

$2,\ldots,\nu$, let p_i be the minimal-degree monic polynomial such that if $d_{m-\eta+i}$ has a zero at a point $\lambda \in \sigma$ of order k then $p_i d_i$ has a zero at λ of order k. Define an $m \times \eta$ matrix polynomial $Q = [q_{ij}]$ by

$$q_{ij} = \begin{cases} 1 & \text{if} \quad i = j \leqslant \eta - \mu, \\ p_i & \text{if} \quad \eta - \mu < i = j \leqslant \nu, \\ 1 & \text{if} \quad i = j - \eta + m > m - \mu, \\ 0 & \text{otherwise,} \end{cases}$$

and put

$$W_2 = W_1 Q.$$

Step 3. For $i = 1,2,\ldots,\mu$, let ϕ_i be the $(m-i-1)$th row of E^{-1}, so that if H has a zero at a point $\lambda \in \sigma$ of the geometric multiplicity κ, then $\{\phi_1,\phi_2,\ldots,\phi_\kappa\}$ is a canonical set of left null functions for H at λ. Find an orthogonal complement Ξ of the row span of P in $(\mathscr{R}^{1\times m},\sigma(A_\pi)\cup\sigma(A_\zeta))$ and functions $\psi_1,\psi_2,\ldots,\psi_\mu \in \Xi$, analytic on $\sigma(A_\pi)\cup\sigma(A_\zeta)$, such that the pair (A_ζ, B_ζ) is left similar to the pair constructed from Taylor coefficients of $\psi_1,\psi_2,\ldots,\psi_\mu$ at the zeros of H in σ. Project every column of W_2 onto $\operatorname{Ker} P$ along Ξ° to get W_3.

Step 4. Find an orthogonal complement Λ of the column span of W_3 in $(\operatorname{Ker} P,\sigma(A_\pi)\cup\sigma(A_\zeta))$, and a basis $\{v_1,v_2,\ldots,v_l\}$ for Λ such that the function $[v_1 \ v_2 \ \cdots \ v_l]$ has neither zeros nor poles on $\sigma(A_\pi) \cup\sigma(A_\zeta)$. Put $W_4 = [W_3 \ v_1 \ v_2 \ \cdots \ v_l]$.

Step 5. Multiply W_4 on the right by a regular rational matrix function Q, without poles or zeros on $\sigma(A_\pi)\cup\sigma(A_\zeta)$, so that the resulting function W has neither poles nor zeros in $\sigma\setminus[\sigma(A_\pi)\cup\sigma(A_\zeta)]$.

Particular steps of this construction are illustrated in [12] with specific examples. It is also shown there that W has (C_π, A_π) as a right pole pair over σ, (A_ζ, B_ζ) as a left null pair over σ, and τ_κ as a left kernel polynomial. It turns out that Γ is the coupling operator for (C_π, A_π) and (A_ζ, B_ζ) as a right pole pair and a left null pair for W over σ.

THEOREM 6.2. *Let $\sigma, H, \tau_s, \tau_\kappa$, and W be as in Construction* 6.1. *Then τ_κ is a left kernel triple for W, and τ_s is a left spectral triple for W over σ.*

Proof. In light of the remark preceding the theorem, it suffices to show that Γ is the coupling matrix for (C_π, A_π) and (A_ζ, B_ζ) viewed as a right pole pair and a left null pair for W over σ. Now, by Theorem 3.3, τ_s is a left σ-spectral triple for W_1. Consequently, since W_2 has (C_π, A_π) as a right pole pair and (A_ζ, B_ζ) as a left null pair over σ (cf. Lemma 4.7 in [12]), and $W_2\mathscr{R}^{\eta\times 1}(\sigma)\subset W_1\mathscr{R}^{m\times 1}(\sigma)$, by Lemma 3.2 above τ_s is a left spectral triple for W_2 over σ. In particular, τ_s is a left spectral triple for W_2 over $\sigma(A_\pi)\cup\sigma(A_\zeta)$.

By the argument in the proof of Lemma 4.9 in [12], W_3 has (C_π, A_π) as a right pole pair and (A_ζ, B_ζ) as a left null pair over $\sigma(A_\pi)\cup\sigma(A_\zeta)$. By Proposition 4.6 in [12], each column ϕ_i of W_3 differs from the corresponding column of W_2 by a function $h_i\in\Xi^\circ$ which is analytic on $\sigma(A_\pi)\cup\sigma(A_\zeta)$. Hence, in view of Theorem 2.7 and Lemma 2.5,

$$W_3\mathscr{R}^{\eta\times 1}(\sigma)=\Big\{C_\pi(z-A_\pi)^{-1}x+h(z): x \text{ is a constant vector and } \sum_{z_0\in\sigma}\operatorname{Res}_{z=z_0}(z-A_\zeta)^{-1}B_\zeta h(z)=\Gamma x\Big\}\cap W_3\mathscr{R}^{\eta\times 1}.$$

By Lemma 5.1, τ_s is a left spectral triple for W_3 over $\sigma(A_\pi)\cup\sigma(A_\zeta)$.

Now W_4 has (C_π, A_π) and (A_ζ, B_ζ) as a right pole pair and a left null pair over σ (cf. Lemma 4.11 in [12]); thus by Lemma 3.2 above, τ_s is a left spectral triple for W_4, and hence also for W, over $\sigma(A_\pi)\cup\sigma(A_\zeta)$. Since W has neither poles nor zeros in $\sigma\setminus[\sigma(A_\pi)\cup\sigma(A_\zeta)]$, τ_s is a left spectral triple for W over σ. ■

REFERENCES

1 J. A. Ball, N. Cohen, and L. Rodman, Zero data and interpolation problems for rectangular matrix polynomials, *Linear and Multilinear Algebra*, to appear.

2 J. A. Ball, N. Cohen, and A. C. M. Ran, Iverse spectral problems for regular improper rational matrix functions, in *Topics in Interpolation Theory of Rational Matrix Functions* (I. Gohberg, Ed.), OT 33, Birkhäuser, Boston, 1988, pp. 123–175.

3 J. A. Ball, I. Gohberg, and L. Rodman, Realization and interpolation of rational matrix functions, in *Topics in Interpolation Theory of Rational Matrix Functions* (I. Gohberg Ed.), OT 33, Birkhäuser, Boston, 1988, pp. 1–72.

4 J. A. Ball, I. Gohberg, and L. Rodman, Two-sided Lagrange-Sylvester interpolation problems for rational matrix functions, in *Proc. Sympos. Pure Math.*, Amer. Math. Soc., vol. 51, Part I, *Amer. Math. Soc.*, 1990, pp. 17–83.

5 J. A. Ball, I. Gohberg, and L. Rodman, Minimal factorization of meromorphic matrix functions in terms of local data, *Integral Equations Operator Theory* 10:309–348 (1987).

6 J. A. Ball, I. Gohberg, and L. Rodman, *Interpolation of Rational Matrix Functions*, OT 45, Birkhäuser, Boston, 1990.

7 J. A. Ball and J. W. Helton, A Beurling-Lax theorem for the Lie group $U(m,n)$ which contains most classical interpolation, *J. Operator Theory* 9:107–142 (1983).

8 J. A. Ball and J. W. Helton, Beurling-Lax representations using classical Lie groups with many applications II: $GL(n,C)$ and Wiener-Hopf factorization, *Integral Equations Operator Theory*, 7:291–309 (1984).

9 J. A. Ball and J. W. Helton, Beurling-Lax representations using classical Lie groups with many applications III: Groups preserving two bilinear forms, *Amer. J. Math*. 108:95–174 (1986).

10 J. A. Ball and J. W. Helton, Beurling-Lax representations using classical Lie groups with many applications IV: $GL(n,R)$, $U^*(2n)$, $SL(n,C)$ and a solvable group, *J. Funct. Anal.* 69:178–206 (1986).

11 J. A. Ball and M. Rakowski, Minimal McMillan degree rational matrix functions with prescribed zero-pole structure, *Linear Algebra Appl.* 137/138:325–349 (1990).

12 J. A. Ball and M. Rakowski, Zero-pole structure of nonregular rational matrix functions, in *Extension and Interpolation of Linear Operators and Matrix Functions* OT47, Birkhäuser, Boston, 1990, pp. 137–193.

13 J. A. Ball and A. C. M. Ran, Global inverse spectral problems for rational matrix functions, *Linear Algebra Appl.* 86:237–282 (1987).

14 J. A. Ball and A. C. M. Ran, Local inverse spectral problems for rational matrix functions, *Integral Equations Operator Theory*, 10:349–415 (1987).

15 T. Chen and B. A. Francis, Spectral and inner-outer factorization of rational matrices, *SIAM J. Matrix Anal. Appl.* 10:1–17 (1989).

16 G. Conte and A. M. Perdon, On the causal factorization problem, *IEEE Trans. Automat. Control*, AC-30 (8):811–813 (Aug. 1985).

17 G. Conte, A. M. Perdon, and B. F. Wyman, Zero/pole structure of linear transfer functions, in *Proceedings of the 24th IEEE Conference on Decision and Control*, Ft. Lauderdale, Fla. Dec. 1985, pp. 529–530.

18 G. D. Forney, Jr., Minimal bases of rational vector spaces, with applications to multivariable linear systems, *SIAM. J. Control*, 13 (3):493–520 (May 1975).

19 I. Gohberg and M. A. Kaashoek, An inverse spectral problem for rational matrix functions and minimal divisibility, *Integral Equations Operator Theory*, 10:437–465 (1987).

20 Y. S. Hung, H^∞ interpolation of rational matrices, *Internat. J. Control*, 48:1659–1713 (1988).

21 N. Jacobson, *Basic Algebra I*, 2nd ed., Freeman, New York, 1985.

22 R. E. Kalman, Algebraic structure of linear dynamic systems I. The module of sigma, *Proc. Nat. Acad. Sci. U.S.A.* 54:1503–1508 (1965).

23 A. F. Monna, *Analyse Non-archimédienne*, Springer, New York, 1970.

24 A. S. Morse, System invariants under feedback and cascade control, in *Mathematical Systems Theory, Proceedings of the International Symposium, Udine, Italy, June 16–27, 1975*, Lecture Notes in Econom. and Math. Systems, Springer-Verlag, New York, 1976, pp. 61–74.

25 B. Sz. Nagy and. C. Foias, *Harmonic Analysis of Operators on Hilbert Space*, North Holland/Amer. Elsevier, New York, 1970.

26 H. Rosenbrock, *State Space and Multivariable Theory*, Wiley, New York, 1970.

27 M. Rakowski, Zero-Pole Interpolation of Nonregular Rational Matrix Functions, Ph.D. Dissertation, Virginia Tech, Blacksburg, Dec. 1989.

28 M. Rosenblum and J. Rovnyak, *Hardy Classes and Operator Theory*, Oxford U. P., New York, 1985.

29 B. Wyman and M. K. Sain, The zero module and essential inverse systems, *IEEE Trans. Circuits and Systems* CAS-28:112–126 (1981).

30 B. F. Wyman, G. Conte, and A. M. Perdon, Local and Global linear system theory, in *Proceedings Internationl Symposium of the Mathematical Theory of Networks and Systems*, 1986, pp. 165–181.

31 B. F. Wyman, M. K. Sain, G. Conte, and A. M. Perdon, On the zeros and poles of a transfer function, *Linear Algebra Appl.* 122/123/124:123–144 (1989).

32 F.-B. Yeh and L.-F. Wei, Inner-outer factorizations of right-invertible real-rational matrices, *Systems Control Lett.* 14:31–36 (1990).

Received 6 May 1990; final manuscript accepted 9 October 1990